MATHEMATICAL TRACTS

ON

Physical Astronomy,

THE FIGURE OF THE EARTH,

PRECESSION and NUTATION,

AND

THE CALCULUS OF VARIATIONS.

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DESIGNED

FOR THE USE OF STUDENTS

IN THE UNIVERSITY.

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PREFACE.

The Work now presented to the Public, consists of four distinct Treatises. Of these, the three first are in some degree connected: the fourth has no relation whatever to the preceding.

The first is entitled, with some impropriety of language, a Treatise on Physical Astronomy; containing, in fact, little more than the most important parts of the Lunar Theory. This the Author has been induced to publish, because an elementary treatise on the problem of Three Bodies is yet wanting in Cambridge. It appeared desirable, that this important subject should be treated in such a manner, as to make it generally accessible to Students of the University, and easy to introduce into the Public Lectures. The entire neglect of the analytical mode of treating Physical Astronomy, cannot be considered otherwise than as a defect in our Mathematical System. The methods of Newton are beautiful, but they have all the imperfection which necessarily accompanies first

attempts: for the explanation of some of the Lunar inequalities, they are hardly sufficient; and for the calculation of most, they are quite inadequate. For other branches of Physical Astronomy, as the Planetary Theory, their inadequacy has never been questioned. Yet in this University, so completely was the attention long confined to the works of Newton, that a few years since, it might justly be said, that we were far behind the rest of the world. The large work of Vince was by no means calculated for general use; the valuable treatise of Professor Woodhouse was the first work on this subject published in Cambridge, whose size and general character placed it within the reach of students, and whose plan was so comprehensive, as to give extensive information on every case of the Problem of Three Bodies. It is scarcely necessary to say, that the short Tract now presented to the Public, is not intended, in the slightest degree, to supersede the Treatise of Professor Woodhouse. But valuable as that Treatise is to the student of talent and acquirements, it is not found to be perfectly adapted for general instruction. The Author of the present Work, has endeavoured to put the Lunar Theory in such a form, that all may be able to comprehend it. By dividing it into Propositions, in the manner which experience shews to be the best adapted for most readers, he hopes

to allure some, who might be deterred by long Chapters and investigations without enunciations. By referring to the Principia, wherever it is possible, he aims to give the subject that interest, which the connexion between different systems generally excites. The present publication he regards as little more than an experiment; he has, therefore, confined himself to the Lunar Theory. The student, to whom the methods of the Lunar Theory are familiar, will have no difficulty in reading the able explanation of the Planetary and succeeding theories, in the latter part of Professor Woodhouse's work.

The second Tract is on the Figure of the Earth, first considering it as homogeneous, and next as heterogeneous. On the Earth's form, supposing it homogeneous, the only treatise generally read in the University, is Maclaurin's Prize Dissertation on the Tides. Though the best parts of that elegant essay apply to the Earth's figure, equally as affected by tides, or by centrifugal force, yet the writer's object has made him treat more particularly of the former. As a treatise on the Figure of the Earth, it is, therefore, imperfect: some additions, also, have been made to this theory by later writers. The first part of the present treatise it is therefore hoped will be found useful. With respect

to the second, it is only necessary to observe, that there is no book commonly used in Cambridge in which Clairaut's theorem is demonstrated; although within a short time more extensive series of pendulum experiments have been made to determine the ellipticity of the Earth, than were ever made before. The admirable work of Clairaut, besides the inconvenience of being written in a foreign language, is loaded with so many speculations, relating little to the subject, that it can never be much used. To supply this defect, the author has endeavoured in the second part to make the theory of the Earth, supposed heterogeneous, accessible to Cambridge students. He has also slightly alluded to the effect of the Earth's ellipticity on the motion of the Moon.

On Precession and Nutation, considered with reference to their physical causes, there is no treatise in use in the University. The propositions of Newton are both imperfect and erroneous; and the necessity for some treatise on this subject is shewn by the circumstance, that most students are familiar with the facts of Precession and Nutation, while very few are acquainted with their causes. This the author has endeavoured to supply: taking Frisi's theorem as the foundation, he has shewn how, from assumed data, the numbers of Precession and Nu-

tation can be calculated, and how, from observations, the Moon's mass and Earth's ellipticity may be found. In this is comprehended the supposition of the Earth's variable density. These three treatises, which might all be included under the title of Physical Astronomy, will enable the aspiring student to comprehend every theorem of importance relating to the form and motions of the Earth and Moon.

The fourth treatise, on the Calculus of Variations, is printed with the others merely because it is too small to be published alone. On this subject, the most beautiful of all the branches of the Differential Calculus, the translation of Lacroix, commonly used in Cambridge, is singularly confused and unintelligible. Professor Woodhouse's interesting treatise on Isoperimetrical Problems, taking the subject in an historical order, is, of course, unfit for the student. Some of the subjects of the preceding tracts are really difficult; but in this theory, which is usually viewed with far greater terror, there is little more difficulty than in the common theory of Maxima and Minima. By adhering rigorously to principles, by exemplifying every formula, and by avoiding the investigation of useless theorems, the author hopes that he has removed many of the difficulties which have been thought to beset this theory.

The author has to apologize for the introduction of an uncommon symbol of integration. For the student who confines himself to the use of differential coefficients, it appeared necessary to employ a symbol which should not require the use of a differential: no confusion, it is imagined, will be occasioned to those who do not adopt that system. The definition of a differential coefficient referred to throughout the work is "the limit of the ratio of the corresponding increments of the function and the independent variable."

The Syndics of the University Press, from the funds of which they have the management, have contributed most liberally to the exper.. of publishing the work: and the author takes this opportunity of publicly acknowledging his obligations.

PHYSICAL ASTRONOMY

INTRODUCTION.

In our succeeding investigations, the following equation will several times occur, and it will therefore be convenient to premise its solution.

As the cases of it which will present themselves are distinguished by some peculiarities, we shall here consider each of them separately.

By the notation $\int_\theta \cos n\theta \,.\, \Theta$, $\int_\theta \cos n\theta \,.\, \cos \overline{m\theta + D}$, &c. we mean what are usually written

$$\int \cos n\theta \,.\, \Theta \,.\, d\theta,\ \int \cos n\theta \,.\, \cos \overline{m\theta + D} \,.\, d\theta,\ \&c.$$

they are the quantities whose differential coefficients, with respect to θ, are $\cos n\theta \,.\, \Theta$, $\cos n\theta \,.\, \cos \overline{m\theta + D}$, &c.

1. Prop. 1. To solve the equation

$$\frac{d^2 u}{d\theta^2} + n^2 u + \Theta = 0;$$

Θ being a function of θ and constants only.

Multiply the equation by $\cos n\theta$,

$$\text{then } \cos n\theta \,.\, \frac{d^2 u}{d\theta^2} + n^2 \cos n\theta \,.\, u + \cos n\theta \,.\, \Theta = 0;$$

integrate the first term by parts;

$$\text{then } \int_\theta \cos n\theta \cdot \frac{d^2u}{d\theta^2} = \cos n\theta \cdot \frac{du}{d\theta} + n\int_\theta \sin n\theta \cdot \frac{du}{d\theta}$$

$$= \cos n\theta \cdot \frac{du}{d\theta} + n \sin n\theta \cdot u - n^2 \int_\theta \cos n\theta \cdot u.$$

Hence, the integral of the whole is

$$\cos n\theta \cdot \frac{du}{d\theta} + n \cdot \sin n\theta \cdot u + \int_\theta \cos n\theta \cdot \Theta = 0,$$

the arbitrary constant being included in the sign of integration. Divide this equation by $\cos^2 n\theta$:

$$\text{then } \frac{\frac{du}{d\theta}}{\cos n\theta} + \frac{n \sin n\theta \cdot u}{\cos^2 n\theta} + \frac{1}{\cos^2 n\theta} \int_\theta \cos n\theta \cdot \Theta = 0:$$

$$\text{integrating, } \frac{u}{\cos n\theta} + \int_\theta \frac{1}{\cos^2 n\theta} \int_\theta \cos n\theta \cdot \Theta = 0,$$

$$\text{or } u = -\cos n\theta \int_\theta \frac{1}{\cos^2 n\theta} \cdot \int_\theta \cos n\theta \cdot \Theta.$$

2. *Case* 1. Let $\Theta = 0$, or $\frac{d^2u}{d\theta^2} + n^2 u = 0 \cdot$

$$\text{then } \cos n\theta \cdot \Theta = 0; \int_\theta \cos n\theta \cdot \Theta = -C;$$

$$\therefore u = \cos n\theta \int_\theta \frac{C}{\cos^2 n\theta} = \cos n\theta \left(\frac{C}{n} \tan n\theta + C'\right)$$

$$= \frac{C}{n} \sin n\theta + C' \cos n\theta.$$

This may be put under the form $A \cos (n\theta + B)$,

$$\text{making } A \cos B = C', \ A \sin B = -\frac{C}{n}.$$

Or under the form $E \sin(n\theta + F)$,

$$\text{making } E\cos F = \frac{C}{n},\quad E\sin F = C':$$

3. *Case* 2. Let $\Theta = -a$;

$$\text{then } \int_\theta \cos n\theta \,.\, \Theta = -\frac{a}{n}\sin n\theta - C;$$

$$u = \cos n\theta \int_\theta \left(\frac{a}{n} \,.\, \frac{\sin n\theta}{\cos^2 n\theta} + \frac{C}{\cos^2 n\theta}\right)$$

$$= \cos n\theta \,.\, \left(\frac{a}{n^2} \,.\, \frac{1}{\cos n\theta} + \frac{C}{n}\tan n\theta + C'\right)$$

$$= \frac{a}{n^2} + \frac{C}{n}\sin n\theta + C'\cos n\theta,$$

$$\text{or } = \frac{a}{n^2} + A\cos(n\theta + B).$$

4. *Case* 3. Let $\Theta = b \,.\, \cos \overline{m\theta + D}$:

$$\text{then } \int_\theta \cos n\theta \,.\, \Theta, \text{ or } b\int_\theta \cos n\theta \,.\, \cos \overline{m\theta + D},$$

integrating by parts,

$$= \frac{b}{n}\sin n\theta \,.\, \cos \overline{m\theta + D} + \frac{mb}{n}\int_\theta \sin n\theta \,.\, \sin \overline{m\theta + D}$$

$$= \frac{b}{n}\sin n\theta \,.\, \cos \overline{m\theta + D} - \frac{mb}{n^2}\cos n\theta \,.\, \sin \overline{m\theta + D}$$

$$+ \frac{m^2 b}{n^2}\int_\theta \cos n\theta \,.\, \cos \overline{m\theta + D}.$$

$$\text{Hence } b\int_\theta \cos n\theta \,.\, \cos \overline{m\theta + D}$$

$$= \frac{b}{m^2 - n^2}\left(m\cos n\theta \,.\, \sin \overline{m\theta + D} - n\sin n\theta \,.\, \cos \overline{m\theta + D}\right) + C.$$

Then

$$u=\cos n\theta \,.\, \frac{b}{m^2-n^2}\int_\theta \left(\frac{m \sin \overline{m\theta+D}}{\cos n\theta}+\frac{n \sin n\theta \,.\, \cos .\, \overline{m\theta+D}}{\cos^2 n\theta}\right)$$

$$+\cos n\theta \int_\theta \,.\, \frac{C}{\cos^2 n\theta}$$

$$=\frac{b}{m^2-n^2} \,.\, \cos \overline{m\theta+D}+\frac{C}{n} \sin n\theta+C' \cos n\theta,$$

$$\text{or } =\frac{b}{m^2-n^2} \,.\, \cos \overline{m\theta+D}+A \cos \overline{n\theta+B}.$$

5. *Case* 4. Let $\Theta=b \cos \overline{n\theta+D}$.

Then $\cos n\theta \,.\, \Theta=\frac{b}{2}(\cos \overline{2n\theta+D}+\cos D)$;

$$\text{its integral } =\frac{b}{2}\left(\frac{\sin \overline{2n\theta+D}}{2n}+\theta \,.\, \cos D\right)-C$$

$$=\frac{b}{2}\left(\frac{\sin 2n\theta \,.\, \cos D}{2n}+\frac{\cos 2n\theta \,.\, \sin D}{2n}+\theta \cos D\right)-C,$$

and

$$u=-\frac{b}{2} \cos n\theta \int_\theta\left(\frac{2 \sin n\theta . \cos n\theta . \cos D}{2n \cos^2 n\theta}+\frac{(2\cos^2 n\theta-1) \sin D}{2n \cos^2 n\theta}+\frac{\theta \cos D}{\cos^2 n\theta}\right)$$

$$+\frac{C}{n} \sin n\theta+C' \cos n\theta$$

$$=-\frac{b}{2} \cos n\theta \int_\theta\left(\cos D \,.\, \overline{\frac{\tan n\theta}{n}+\frac{\theta}{\cos^2 n\theta}}+\sin D \,.\, \overline{\frac{1}{n}-\frac{1}{2n . \cos^2 n\theta}}\right)$$

$$+\frac{C}{n} \sin n\theta+C' \cos n\theta$$

$$=-\frac{b}{2} \cos n\theta\left(\cos D \,.\, \frac{\theta \,.\, \tan n\theta}{n}+\sin D \,.\, \overline{\frac{\theta}{n}-\frac{1}{2n^2} \tan n\theta}\right)$$

$$+\frac{C}{n} \sin n\theta+C' \cos n\theta$$

$$=-\frac{b}{2}\left(\frac{\theta}{n} \sin \overline{n\theta+D}-\frac{1}{2n^2} \sin n\theta \,.\, \sin D\right)+\frac{C}{n} \sin n\theta+C' \cos n\theta.$$

Or, if we include the coefficient $\frac{b}{4n^2}\sin D$ under the arbitrary constant $\frac{C}{n}$,

$$u = -\frac{b}{2}\cdot\frac{\theta}{n}\sin\overline{n\theta + D} + \frac{C}{n}\sin n\theta + C'\cos n\theta.$$

6. *Remarks.* If Θ consisted of several terms, the expression for u would contain one term corresponding to each. The part which depends upon the arbitrary constants, is entirely independent of Θ. The process above having shewn what is the form of the expression of u, we may sometimes solve the equation with greater ease, by assuming an expression with indeterminate coefficients. Thus, if we had the equation

$$\frac{d^2u}{d\theta^2} + n^2u + a + b\cos\overline{m\theta + B} + p\cos q\theta + Q = 0,$$

we might assume

$$u = -\frac{a}{n^2} + A\cos(n\theta + C) + D\cos\overline{m\theta + B} + E\cos\overline{q\theta + Q},$$

and, substituting this series in the equation, determine the values of D and E.

7. When m does not differ much from n, it appears from (4), that the coefficient of $\cos\overline{m\theta + D}$, in the expression for u, will be much greater than that in the original equation. This remark we shall find to be very important. The solution in the 4th case assumes a form different from any of the others: its peculiarity will materially affect our future operations.

MOTION OF TWO BODIES.

8. If the Sun were supposed to be at rest, the motion of a planet about it might be found by the formulæ for central

forces. In the equation $\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0$, (Whewell's *Dynamics*, Art. 18.) we must put for P the attraction of the Sun on the planet, and by solving the equation, we should find u in terms of θ, and the form of the orbit which the planet describes would then be known.

9. In the actual case of the Sun and a planet, these bodies move about their common centre of gravity. But their relative motion will be the same as if we suppose the Sun to be at rest, provided we add to the accelerating forces which really act on the planet, another force equal and opposite to that which acts on the Sun. For if the same accelerating forces be supposed to act on both, since the absolute motion which it communicates to both is the same, and in the same direction, their relative motion will be the same as if that force did not act: and if that force be equal and opposite to the force really acting on the Sun, the Sun will be at rest. In the same way, if we add to the forces acting on the Sun, a force equal and opposite to that acting on the planet, the planet will be at rest, and the relative motions will be unaltered. We shall generally make the latter supposition.

10. PROP. 2. The orbit which the Sun appears to describe about a planet is a conic section.

Let M = mass of Sun (estimated by the accelerating force which its attraction produces at distance 1), M' = that of the planet: let their distance = r. The accelerating force on the Sun, on the law of gravitation, $= \frac{M'}{r^2}$: that on the planet $= \frac{M}{r^2}$: if then we suppose this force applied in the opposite direction to the Sun, the whole accelerating force on the Sun, supposing the planet at rest, $= \frac{M+M'}{r^2} = (M+M')\,u^2$, if $u = \frac{1}{r}$. Substituting this for P in the equation above,

$$\frac{d^2u}{d\theta^2} + u - \frac{M+M'}{h^2} = 0,$$

the solution of which, by (3), is

$$u = \frac{M+M'}{h^2} + A \cos \overline{\theta + B},$$

$$\text{or } r = \frac{1}{\frac{M+M'}{h^2} + A \cos \overline{\theta + B}},$$

which is the general polar equation to the conic sections, the focus being the pole.

11. The conic section which a planet appears to describe about the Sun, or the Sun about a planet, is an ellipse. Let a and e be the semi-axis-major and eccentricity, then

$$r = \frac{1}{\frac{1}{a(1-e^2)} + \frac{e}{a(1-e^2)} \cos \overline{\theta + B}}.$$

Comparing this with the expression above,

$$\frac{M + M'}{h^2} = \frac{1}{a(1-e^2)},$$

$$\text{or } h = \sqrt{a(1-e^2)(M+M')}$$

12. Prop. 3. To find the time of describing any part of the ellipse, or to express nt (the mean anomaly) in terms of θ, (the true anomaly).

By Whewell's *Dynamics*, Art. 16, $\frac{dt}{d\theta} = \frac{r^2}{h} =$ (in the present instance), $\frac{a^{\frac{3}{2}}.(1-e^2)^{\frac{3}{2}}}{\sqrt{M+M'}} . \frac{1}{(1+e \cos \overline{\theta + B})^2}$. The most convenient form into which this can be expanded, is a series of cosines of multiple arcs, as

$$A + C \cos \overline{\theta + B} + D \cos 2.\overline{\theta + B} + E \cos 3.\overline{\theta + B} + \&c.$$

To effect this, we shall first expand $\frac{1}{1+e\cos\overline{\theta+B}}$ in a similar series.

13. If for $\cos\overline{\theta+B}$, we put $\frac{x+\frac{1}{x}}{2}$ $(x=\epsilon^{\overline{\theta+B}\sqrt{-1}})$, we have

$$\frac{1}{1+e\cos\overline{\theta+B}}=\frac{1}{1+\frac{e}{2}\left(x+\frac{1}{x}\right)}, \text{ which will} = \frac{1}{(\alpha+\beta x)\left(\alpha+\frac{\beta}{x}\right)},$$

$$\text{if } \alpha^2+\beta^2=1, \quad \alpha\beta=\frac{e}{2}.$$

From these equations $\alpha=\frac{1}{2}(\sqrt{1+e}+\sqrt{1-e})$:

$$\beta=\frac{1}{2}(\sqrt{1+e}-\sqrt{1-e});$$

$$\therefore \alpha^2=\frac{1+\sqrt{1-e^2}}{2}; \quad \frac{\beta}{\alpha}=\frac{e}{1+\sqrt{1-e^2}}; \text{ let this} = \lambda.$$

Then $$\frac{1}{1+e\cos\overline{\theta+B}}=\frac{2}{1+\sqrt{1-e^2}}\cdot\frac{1}{(1+\lambda x)\left(1+\frac{\lambda}{x}\right)}$$

$$=\frac{2}{1+\sqrt{1-e^2}}\cdot\frac{x}{(1+\lambda x)(x+\lambda)}$$

$$=\frac{2}{1+\sqrt{1-e^2}}\cdot\frac{1}{1-\lambda^2}\cdot\left(\frac{1}{1+\lambda x}-\frac{\lambda}{x+\lambda}\right)$$

$$=\frac{2}{1+\sqrt{1-e^2}}\cdot\frac{1}{1-\lambda^2}\cdot\left\{\frac{1}{1+\lambda x}-\frac{\frac{\lambda}{x}}{1+\frac{\lambda}{x}}\right\}.$$

Expanding these fractions, and observing that

$$x + \frac{1}{x} = 2\cos\overline{\theta+B},\quad x^2 + \frac{1}{x^2} = 2\cos 2.\overline{\theta+B},\ \&c.,$$

$$\frac{1}{1+e\cos\overline{\theta+B}} = \frac{2}{1+\sqrt{1-e^2}}\cdot\frac{1}{1-\lambda^2}\times$$

$$(1 - 2\lambda\cos\overline{\theta+B} + 2\lambda^2.\cos 2.\overline{\theta+B} - 2\lambda^3.\cos 3.\overline{\theta+B} + \&c.)$$

$$\text{But } 1-\lambda^2 = \frac{2\sqrt{1-e^2}}{1+\sqrt{1-e^2}};\ \text{hence}$$

$$\frac{1}{1+e\cos\overline{\theta+B}} = \frac{1}{\sqrt{1-e^2}}.(1-2\lambda.\cos\overline{\theta+B}+2\lambda^2.\cos 2.\overline{\theta+B}-\&c.)$$

14. Now $\dfrac{1}{(1+e\cos\overline{\theta+B})^2} = \dfrac{d}{de}\left(\dfrac{e}{1+e\cos\overline{\theta+B}}\right)$:

observing, then, that

$$\frac{d}{de}\left(\frac{e}{\sqrt{1-e^2}}.\lambda^p\right) = \frac{e^p(1+p\sqrt{1-e^2})}{(1+\sqrt{1-e^2})^p.(1-e^2)^{\frac{3}{2}}},$$

we have

$$\frac{1}{(1+e\cos\overline{\theta+B})^2} = \frac{1}{(1-e^2)^{\frac{3}{2}}}.\left\{1 - 2.\frac{e(1+\sqrt{1-e^2})}{1+\sqrt{1-e^2}}\cos\overline{\theta+B}\right.$$

$$+2.\frac{e^2(1+2\sqrt{1-e^2})}{(1+\sqrt{1-e^2})^2}\cos 2.\overline{\theta+B}$$

$$\left.-2\frac{e^3(1+3\sqrt{1-e^2})}{(1+\sqrt{1-e^2})^3}\cos 3.\overline{\theta+B}+\&c.\right\}.$$

15. Hence (12), $\dfrac{dt}{d\theta} = \dfrac{a^{\frac{3}{2}}}{\sqrt{M+M'}}\times$

$$\left\{1 - 2e \cos \overline{\theta + B} + \frac{2e^2(1 + 2\sqrt{1-e^2})}{(1+\sqrt{1-e^2})^2} \cos 2 . \overline{\theta + B} - \&c.\right\}.$$

Integrating and correcting, so as to make $t=0$, when $\theta + B = 0$, and putting $\frac{\sqrt{M+M'}}{a^{\frac{3}{2}}} = n$,

$$nt = \overline{\theta + B} - 2e \sin \overline{\theta + B} + \frac{e^2(1+2\sqrt{1-e^2})}{(1+\sqrt{1-e^2})^2} \sin 2 . \overline{\theta + B} - \&c.$$

$$\pm \frac{2e^p . (1 + p\sqrt{1-e^2})}{p(1+\sqrt{1-e^2})^p} \sin p . \overline{\theta + B} \mp \&c.$$

For a whole revolution, suppose θ increased by 2π: then if T be the periodic time,

$$nT = 2\pi, \text{ or } T = \frac{2\pi}{n} = \frac{2\pi . a^{\frac{3}{2}}}{\sqrt{M+M'}}.$$

16. The term *anomaly* is used generally to denote the angular distance of that body which is supposed to move, from its apse. The true anomaly, then, in the present case, is $\theta + B$. But nt is an angle which is proportional to the time, and is that through which, if the body had moved uniformly with such an angular velocity, that it would have performed a revolution in the same time in which it does perform it, it would have moved in the time t. It is therefore called the *mean* anomaly.

17. If we expand these coefficients as far as e^3,

$$nt = \overline{\theta + B} - 2e \sin \overline{\theta + B} + \tfrac{3}{4} e^2 \sin 2 . \overline{\theta + B} - \frac{e^3}{3} \sin 3 . \overline{\theta + B}.$$

18. Prop. 4. To find $\overline{\theta + B}$ in terms of nt, or the true anomaly in terms of the mean. This must be done by Lagrange's theorem. If $y = z + x . \phi(y)$, then

$$y = z + \phi(z) . \frac{x}{1} + \frac{d}{dz}\{\overline{\phi(z)}|^2\} . \frac{x^2}{1.2} + \frac{d^2}{dz^2}\{\overline{\phi(z)}|^3\} . \frac{x^3}{1.2.3} + \&c.$$

Here $y = z + e\left(2\sin y - \frac{3e}{4}\sin 2y + \frac{e^2}{3}\sin 3y\right)$;

$$\therefore\ \phi(z) = 2\sin z - \frac{3e}{4}\sin 2z + \frac{e^2}{3}\sin 3z;$$

$$\therefore\ \overline{\phi(z)}|^2 = \left(2\sin z - \frac{3e}{4}\sin 2z\right)^2$$

$$= 2 - 2\cos 2z - \frac{3e}{2}\cos z + \frac{3e}{2}\cos 3z;$$

$$\therefore\ \frac{d}{dz}\left\{\overline{\phi(z)}|^2\right\} = 4\sin 2z + \frac{3e}{2}\sin z - \frac{9e}{2}\,.\,\sin 3z,$$

$$\overline{\phi(z)}|^3 = 8\sin^3 z = 6\sin z - 2\sin 3z;$$

$$\therefore\ \frac{d^2}{dz^2}\left\{\overline{\phi(z)}|^3\right\} = -6\sin z + 18\sin 3z;$$

$$\therefore\ y = z + \left(2\sin z - \frac{3e}{4}\sin 2z + \frac{e^2}{3}\sin 3z\right)\frac{e}{1}$$

$$+ \left(4\sin 2z + \frac{3e}{2}\sin z - \frac{9e}{2}\,.\,\sin 3z\right)\frac{e^2}{2}$$

$$+ (18\sin 3z - 6\sin z)\frac{e^3}{6} = z + \left(2e - \frac{e^3}{4}\right)\sin z$$

$$+ \frac{5e^2}{4}\sin 2z + \frac{13}{12}e^3\,.\,\sin 3z, \text{ as far as } e^3.$$

$$\text{Or } \theta + B = nt + \left(2e - \frac{e^3}{4}\right)\sin nt$$

$$+ \frac{5e^2}{4}\sin 2nt + \frac{13}{12}e^3\sin 3nt + \&c.$$

19. The mean anomaly, then, is that part of the true anomaly, which is independent of periodical terms, as sines or cosines. This is the usual signification of the word *mean*, in Astronomy.

LUNAR THEORY.

20. If the Earth, in its revolution round the Sun, were unaccompanied by any other body, it would accurately describe an ellipse. By the attraction of the Moon, the orbit will be altered: to assist us in the discovery of the orbit really described, the following theorem will be useful.

21. PROP. 4. The centre of gravity of the Earth and Moon describes about the Sun, very nearly, an ellipse in one plane, and the area passed over by its radius vector is very nearly proportional to the time.

Let E and M, (fig. 1.) be the Earth and Moon, m' the Sun; G the center of gravity of the Earth and Moon; join $m'E$, $m'G$, $m'M$; and draw EH, MK, perpendicular to $m'G$; let $m'G = r'$, $EM = r$, $m'E = y$, $\angle\, m'GM = \omega$. Now the accelerating force of m' on E in the direction Em' is $\dfrac{m'}{y^2}$; therefore the moving force in that direction $= \dfrac{m'.E}{y^2}$; therefore the moving force in direction parallel to Gm' is $\dfrac{m'.E}{y^2} \times \dfrac{m'H}{y}$

$$= \frac{m'.E}{y^2} \cdot \frac{r' + GE.\cos\omega}{y}$$

$$= \frac{m'.E\,(r' + GE.\cos\omega)}{y^3}.$$

Similarly, the moving force on M in direction parallel to Gm'

$$= \frac{m'.M}{y'^2} \cdot \frac{m'K}{y'}$$

$$= \frac{m'.M\,(r' - GM.\cos\omega)}{y'^3};$$

therefore the accelerating force on the center of gravity in the direction Gm' is

$$\frac{m'}{M+m}\left\{\frac{E\,(r'+GE\cos\omega)}{y^3}+\frac{M\,(r'-GM\cos\omega)}{y'^3}\right\}.$$

And the moving force of m' on E in direction perpendicular to Gm' is $\dfrac{m'.E}{y^2}.\dfrac{EH}{y}$

$$=\frac{m'.E}{y^2}.\frac{GE\sin\omega}{y}$$

$$=\frac{m'.E}{y^3}\,GE.\sin\omega:$$

that on $M=-\dfrac{m'.M}{y'^2}.\dfrac{MK}{y'}$

$$=-\frac{m'.M}{y'^3}\,GM.\sin\omega;$$

therefore the accelerating force on the centre of gravity in direction perpendicular to Gm' is

$$\frac{m'}{M+E}.\left(\frac{E.GE.\sin\omega}{y}-\frac{M.GM.\sin\omega}{y'^3}\right).$$

22. Now $\dfrac{1}{y^3}=\dfrac{}{(r'^2+2r'.GE.\cos\omega+\overline{GE}^2)^{\frac{3}{2}}}$

$$=\left(\text{putting for }\cos^2\omega\text{ its equal }\frac{1+\cos 2\omega}{2}\right)$$

$$\frac{1}{r'^3}\left\{1-3\,\frac{GE}{r'}\cos\omega+\frac{GE^2}{r'^2}\left(\frac{9}{4}+\frac{15}{4}\cos 2.\omega\right)+\&c.\right\}$$

$$\therefore\ \frac{r'+GE.\cos\omega}{y^3}=\frac{1}{r'^2}\left\{1-2.\frac{GE}{r'}\cos\omega\right.$$

$$\left.+\frac{GE^2}{r'^2}\left(\frac{3}{4}+\frac{9}{4}\cos 2.\omega\right)+\&c.\right\}.$$

Similarly, $$\frac{r' - GM\cos\omega}{y'^3} = \frac{1}{r'^2}\left\{1 + 2\cdot\frac{GM}{r'}\cos\omega + \frac{GM^2}{r'^2}\left(\frac{3}{4} + \frac{9}{4}\cos 2\,.\,\omega\right) + \&c.\right\}.$$

$$\text{But } GE = \frac{M}{E+M}r;\quad GM = \frac{E}{E+M}r;$$

hence, accelerating force on center of gravity in direction Gm'

$$= \frac{m'}{r'^2}\left\{1 + \frac{EM^2 + ME^2}{(E+M)^3}\cdot\frac{r^2}{r'^2}\left(\frac{9}{4} + \frac{15}{4}\cos 2\omega\right) + \&c.\right\}$$

$$= \frac{m'}{r'^2}\left\{1 + \frac{ME}{(E+M)^2}\cdot\frac{r^2}{r'^2}\left(\frac{9}{4} + \frac{15}{4}\cos 2\,.\,\omega\right) + \&c.\right\}$$

Now this differs from $\frac{m'}{r'^2}$ only by a quantity which is multiplied by $\frac{r^2}{r'^2}$, and which, in the lunar theory, $\left(\text{where } \frac{r}{r'} = \frac{1}{400}\text{ nearly}\right)$ is quite insensible. In the same manner, we find the accelerating force perpendicular to Gm'

$$= -\frac{m'}{r'^2}\cdot\frac{3}{2}\cdot\frac{EM^2 + ME^2}{(E+M)^3}\cdot\frac{r^2}{r'^2}\sin 2\omega,$$

$$\text{or } -\frac{m'}{r'^2}\cdot\frac{3}{2}\cdot\frac{ME}{(M+E)^2}\cdot\frac{r^2}{r'^2}\cdot\sin 2\,.\,\omega,$$

which, for the same reason, is too small to be perceptible in its effects. Hence, the center of gravity moves, very nearly, in the same manner in which a body would move placed at G. Similarly it appears, that the force on the Sun, and the motion of the Sun are the same, as if a mass $= E + M$ were collected at G; therefore the relative motion of the center of gravity about the Sun, is the same as that of a mass $= E + M$; that is, it will very nearly describe an ellipse in one plane, making the areas proportional to the times.

23. Cor. The Sun's apparent longitude, therefore, is not that found by the elliptic theory, for that is his longitude as seen from G; but must be found by adding to the longitude so found, the angle $Em'G$. Now sin $Em'G$ or $Em'G$, (since it is a small angle, never exceeding $10''$) $= \frac{EG}{y'}$ sin EGm' $= \frac{M}{E+M} \cdot \frac{r}{r'}$ sin ω very nearly: and, since the orbits of the Earth and Moon are nearly circular, this angle varies as sin ω very nearly. And if the Moon be above the plane of the ecliptic, the Earth will be below it, and the Sun will appear to have a latitude, which can be calculated from the latitude of the Moon.

24. If the Sun did not attract the Earth and Moon, or if it attracted them equally, their relative motions would not be disturbed, and the Moon would accurately describe an ellipse about the Earth. But the Sun attracts them unequally, and in different directions; so that not only is the force altered in the direction of the radius vector, but a force also acts perpendicular to it. And as the Moon's orbit is inclined to the ecliptic, the disturbing force draws the Moon from the plane in which she is moving, and thus the plane of her orbit is perpetually changing. There appears to be no better mode of estimating the disturbing force, than by resolving it into three parts, one in the direction of the projection of the radius vector on the ecliptic, another in the plane of the ecliptic, perpendicular to this projection, and a third perpendicular to the plane of the ecliptic.

25. Prop. 5. To find the resolved part of the Sun's disturbing force on the Moon, in the direction of the projection of the radius vector on the ecliptic.

Let E, M, m', (fig. 2.), be the Earth, Moon, and Sun: G the center of gravity of the Earth and Moon, which by Prop. 4., describes an ellipse in one plane about the Sun, or about which the Sun appears to describe an ellipse in one plane: draw MB, EA, perpendicular to the plane of the ecliptic; join $m'M$, $m'B$, BGA: let $m'G = r'$, $AB = \rho$, $EM = r$, $m'M = y'$, tan $MGB = s$. Then AB is the projection of EM on the

plane of the ecliptic. The force of m' upon M is $\frac{m'}{y'^3}$ in the direction Mm', which is equivalent to $\frac{m'}{y'^2} \cdot \frac{MG}{y'}$ in direction MG, and $\frac{m'}{y'^2} \cdot \frac{Gm'}{y'}$ in direction parallel to Gm'.

Resolving the force $\frac{m' . MG}{y'^3}$ into one parallel to MB, and one parallel to BG,

$$\text{the latter} = \frac{m' . MG}{y'^3} \times \cos MGB = \frac{m' . MG}{y'^3 \sqrt{1+s^2}}:$$

and resolving the force $\frac{m' . Gm'}{y'^3}$ into one parallel to BG, and another perpendicular to BG, in the plane of the ecliptic,

$$\text{the former} = -\frac{m' . Gm'}{y'^3} . \cos m'GB.$$

Let θ be the longitude of M, seen from G; θ' the longitude of m': then $\angle m'GB = \theta - \theta'$: and the part of $\frac{m'.Gm'}{y'^3}$ parallel to BG

$$= -\frac{m'.Gm'}{y'^3} \cos \overline{\theta - \theta'}.$$

Hence, the whole force on M in the direction BG, produced by the Sun's attraction, is

$$m'\left(\frac{MG}{y'^3 \sqrt{1+s^2}} - \frac{Gm'}{y'^3} \cos \overline{\theta - \theta'}\right).$$

Similarly, the whole force on E, estimated in the same direction, is

$$m'\left(-\frac{EG}{y^3 \sqrt{1+s^2}} - \frac{Gm'}{y^3} \cos \overline{\theta - \theta'}\right).$$

If, then, in the same manner as in (9), we suppose this force applied to M in the opposite direction, we have, for the whole disturbing force on M, in direction of the projection of the radius vector,

$$m'\left\{\frac{MG}{y'^3\sqrt{1+s^2}} + \frac{EG}{y^3\sqrt{1+s^2}} - Gm'\cos\overline{\theta-\theta'}\left(\frac{1}{y'^3}-\frac{1}{y^3}\right)\right\}.$$

26. Prop. 6. To find the resolved part of the disturbing force, which is parallel to the plane of the ecliptic, and perpendicular to the projection of the radius vector.

The only force which acts in this direction on M is the resolved part of the force $\frac{m'.Gm'}{y'^3}$ acting parallel to Gm', which

$$\frac{m'.Gm'}{y'^3} \sin\overline{\theta-\theta'},$$

(if we suppose the Moon to move so that the angle θ increases, and if we consider the force as positive when it tends to accelerate the Moon's motion). And the only force on E, in the same direction, is

$$-\frac{m'.Gm'}{y^3}\sin\overline{\theta-\theta'}.$$

Supposing this applied to M in the opposite direction, the whole disturbing force, perpendicular to the projection of the radius vector

$$= -m'.Gm'.\sin\overline{\theta-\theta'}\left(\frac{1}{y'^3}-\frac{1}{y^3}\right).$$

27. Prop. 7. To find the resolved part of the disturbing force, which is perpendicular to the plane of the ecliptic.

The only force on M, perpendicular to the plane of the ecliptic, is the resolved part of the force in MG. It is, therefore.

$$= \frac{m'.MG}{y'^3} . \sin MGB$$

$$= \frac{m'.MG}{y'^3} . \frac{s}{\sqrt{1+s^2}}.$$

The force on E, in the same direction, is

$$= \frac{m'.EG}{y^3} . \frac{s}{\sqrt{1+s^2}}.$$

Applying this to M in the opposite direction, we have, for the whole disturbing force perpendicular to the plane of the ecliptic,

$$m'. \frac{s}{\sqrt{1+s^2}} \left(\frac{MG}{y'^3} + \frac{EG}{y^3}\right).$$

28. Prop. 7. To find the whole force upon M in these directions: or to find P, T, and S.

Besides the disturbing forces, we must also find the forces resulting from the mutual attraction of E and M. The attraction of E upon $M = \frac{E}{r^2}$: that of m upon $E = \frac{M}{r^2}$, in the opposite direction: applying the latter to M with its direction changed, we have, for the whole force on M, $\frac{E+M}{r^2}$ acting in the direction ME. The resolved part of this, in the direction of the projection of the radius vector, is $\frac{E+M}{r^2} \times \cos MGB$

$$= \frac{E+M}{\rho^2 (1+s^2)^{\frac{3}{2}}};$$

the resolved part in the plane of the ecliptic, perpendicular to this projection, $=0$: the resolved part, perpendicular to the plane of the ecliptic,

$$= \frac{(E+M)\, s}{\rho^2 (1+s^2)^{\frac{3}{2}}}.$$

If, then, we put P, T, and S, for the whole forces on M, parallel to the projection of the radius, perpendicular to the projection of the radius, and perpendicular to the ecliptic, supposing E at rest, we have

$$P = \frac{E+M}{\rho^2 (1+s^2)^{\frac{3}{2}}}$$

$$+ m' \left\{ \frac{1}{\sqrt{1+s^2}} \left(\frac{MG}{y'^3} + \frac{EG}{y^3} \right) - r' . \cos(\theta - \theta') \left(\frac{1}{y'^3} - \frac{1}{y^3} \right) \right\},$$

$$T = -m' . r' . \sin(\theta - \theta') . \left(\frac{1}{y'^3} - \frac{1}{y^3} \right),$$

$$S = \frac{(E+M)\, s}{\rho^2 (1+s^2)^{\frac{3}{2}}} + \frac{m\, s}{\sqrt{1+s^2}} \left(\frac{MG}{y'^3} + \frac{EG}{y^3} \right).$$

And $MG = \dfrac{E}{E+M} . \rho \sqrt{1+s^2}$; $EG = \dfrac{M}{E+M} \rho \sqrt{1+s^2}$.

29. We have now the values of the forces upon M in three directions, considering E as fixed. We must, therefore, investigate the differential equations, for the motion of a body about a fixed centre, acted on by forces in these directions.

30. Prop. 9. To find the differential equations for the motion of M about the fixed centre E.

Let $E♈$ be a straight line in the plane of the ecliptic, drawn from E towards the first point of Aries: Mb perpendicular to the ecliptic: then, $♈ Eb$ = Moon's longitude $= \theta$. Also, if x, y, and z be rectangular co-ordinates (z being perpendicular to the plane of the ecliptic, and x measured on the line drawn towards the first point of Aries),

$$x = \rho \cos\theta, \quad y = \rho \sin\theta, \quad z = \rho s.$$

And X, or the force in direction of x, $= - P \cos\theta - T \sin\theta$,

Y . . . . $= - P \sin\theta + T \cos\theta$.

Z . . . . $= - S$.

Hence the equations of motion

$$\frac{d^2x}{dt^2} = X,\ \frac{d^2y}{dt^2} = Y,\ \frac{d^2z}{dt^2} = Z,$$

are changed into the following,

$$\frac{d^2x}{dt^2} = -P\frac{x}{\rho} - T\frac{y}{\rho},$$

$$\frac{d^2y}{dt^2} = -P\frac{y}{\rho} + T\frac{x}{\rho},$$

$$\frac{d^2z}{dt^2} = -S.$$

Hence, $x\dfrac{d^2x}{dt^2} + y\dfrac{d^2y}{dt^2} = -P\dfrac{x^2+y^2}{\rho} = -P\rho.$

But $x^2+y^2=\rho^2$, $x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = \rho\dfrac{d\rho}{dt}$,

$$\therefore\ x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + y\,.\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = \rho\frac{d^2\rho}{dt} + \left(\frac{d\rho}{dt}\right)^2.$$

Hence, since $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \left(\dfrac{d\rho}{dt}\right)^2 + \rho^2.\left(\dfrac{d\theta}{dt}\right)^2$,

$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} = \rho\frac{d^2\rho}{dt^2} - \rho^2\left(\frac{d\theta}{dt}\right)^2,$$

and, therefore, $\rho\dfrac{d^2\rho}{dt^2} - \rho^2\left(\dfrac{d\theta}{dt}\right)^2 = -P\rho,$

$$\text{or}\quad \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2 = -P\ldots\ldots(a).$$

Also $x\dfrac{d^2y}{dt^2} - y\dfrac{d^2x}{dt^2} = T\dfrac{x^2+y^2}{\rho} = T\rho.$

But $x\dfrac{d^2y}{dt^2} - y\dfrac{d^2x}{dt^2} = \dfrac{d}{dt}\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right) = \dfrac{d}{dt}\left(\rho^2.\dfrac{d\theta}{dt}\right);$

$$\therefore \frac{d}{dt}\left(\rho^2 . \frac{d\theta}{dt}\right) = T\rho \ldots\ldots\ldots\ldots (b).$$

$$\text{And } \frac{d^2(\rho s)}{dt^2} = -S \ldots\ldots\ldots (c).$$

31. These equations (*a*), (*b*), (*c*), appear to be the simplest equations to which the motion of a point can be reduced. In their present form, however, it is not possible to integrate them: we must obtain equations between ρ, θ, and s, independent of t. This is the object of the next proposition.

32. Prop. 10. To eliminate t from the differential equations.

Since, in their present form, t is the independent variable, we must take some other quantity for the independent variable. Let this be θ. Now by equation (*b*), $\frac{d}{dt}\left(\rho^2 . \frac{d\theta}{dt}\right) = T\rho$;

multiplying each side by $\rho^2 \frac{d\theta}{dt}$,

$$\rho^2 \frac{d\theta}{dt} . \frac{d}{dt}\left(\rho^2 \frac{d\theta}{dt}\right) = T\rho^3 \frac{d\theta}{dt};$$

$$\therefore \frac{1}{2} . \frac{d}{dt}\left(\rho^2 . \frac{d\theta}{dt}\right)^2 = T\rho^3 . \frac{d\theta}{dt};$$

$$\therefore \frac{1}{2} . \frac{d}{d\theta}\left(\rho^2 . \frac{d\theta}{dt}\right)^2 = T\rho^3;$$

$$\therefore \left(\rho^2 \frac{d\theta}{dt}\right)^2 = h^2 + 2\int_\theta T\rho^3,$$

h^2 being a constant quantity;

$$\text{hence } \rho^2 \frac{d\theta}{dt} = \sqrt{h^2 + 2\int_\theta T\rho^3}, \text{ and } \frac{dt}{d\theta} = \frac{\rho^2}{\sqrt{h^2 + 2\int_\theta T\rho^3}}.$$

Now, to transform equation (a), we observe that $\frac{d\rho}{dt} = \frac{\frac{d\rho}{d\theta}}{\frac{dt}{d\theta}}$: let this $= p$:

$$\text{then } \frac{d^2\rho}{dt^2} = \frac{dp}{dt} = \frac{\frac{dp}{d\theta}}{\frac{dt}{d\theta}}$$

$$= \frac{1}{\frac{dt}{d\theta}} \cdot \left\{ \frac{\frac{d^2\rho}{d\theta^2}}{\frac{dt}{d\theta}} - \frac{\frac{d\rho}{d\theta} \cdot \frac{d^2t}{d\theta^2}}{\left(\frac{dt}{d\theta}\right)^2} \right\} = \frac{\frac{d^2\rho}{d\theta^2}}{\left(\frac{dt}{d\theta}\right)^2} - \frac{\frac{d\rho}{d\theta} \cdot \frac{d^2t}{d\theta^2}}{\left(\frac{dt}{d\theta}\right)^3}.$$

$$\text{Now } \frac{1}{\left(\frac{dt}{d\theta}\right)^2} = \frac{h^2}{\rho^4} + \frac{2}{\rho^4}\int_\theta T\rho^3:$$

$$\text{differentiating, } -\frac{2\frac{d^2t}{d\theta^2}}{\left(\frac{dt}{d\theta}\right)^3} = -\frac{4\frac{d\rho}{d\theta}}{\rho^5}(h^2 + 2\int_\theta T\rho^3) + \frac{2T}{\rho};$$

$$\text{hence, } \frac{d^2\rho}{dt^2} = \frac{d^2\rho}{d\theta^2} \cdot \left(\frac{h^2}{\rho^4} + \frac{2}{\rho^4}\int_\theta T\rho^3\right)$$

$$- \frac{2\left(\frac{d\rho}{d\theta}\right)^2}{\rho^5}(h^2 + 2\int_\theta T\rho^3) + \frac{T}{\rho}\frac{d\rho}{d\theta}$$

$$= \left\{ \frac{\frac{d^2\rho}{d\theta^2}}{\rho^2} - \frac{2\left(\frac{d\rho}{d\theta}\right)^2}{\rho^3} \right\} \cdot \left(\frac{h^2 + 2\int_\theta T\rho^3}{\rho^2}\right) + \frac{T\frac{d\rho}{d\theta}}{\rho}.$$

$$\text{Also } \left(\frac{d\theta}{dt}\right)^2 = \frac{1}{\left(\frac{dt}{d\theta}\right)^2} = \frac{h^2 + 2\int_\theta T\rho^3}{\rho^4};$$

$$\therefore \rho\left(\frac{d\theta}{dt}\right)^2 = \frac{h^2+2\int_\theta T\rho^3}{\rho^3}.$$

Substituting these values, the equation (a) becomes

$$\left\{\frac{\frac{d^2\rho}{d\theta^2}}{\rho^2} - \frac{2\left(\frac{d\rho}{d\theta}\right)^2}{\rho^3} - \frac{1}{\rho}\right\}\frac{h^2+2\int_\theta T\rho^3}{\rho^2} = -P - \frac{T\frac{d\rho}{d\theta}}{\rho}.$$

$$\text{If } u = \frac{1}{\rho}, \quad \frac{du}{d\theta} = -\frac{\frac{d\rho}{d\theta}}{\rho^2}, \quad \frac{d^2u}{d\theta^2} = -\frac{\frac{d^2\rho}{d\theta^2}}{\rho^2} + \frac{2\left(\frac{d\rho}{d\theta}\right)^2}{\rho^3}$$

By this substitution, the equation becomes

$$\left(-\frac{d^2u}{d\theta^2} - u\right).\left(h^2+2\int_\theta \frac{T}{u^3}\right)u^2 = -P + T.\frac{\frac{du}{d\theta}}{u},$$

$$\text{or } \frac{d^2u}{d\theta^2} + u - \frac{\frac{P}{u^2} - \frac{T}{u^3}.\frac{du}{d\theta}}{h^2+2\int_\theta \frac{T}{u^3}} = 0. \qquad (d).$$

33. In the same manner, in equation (c), instead of $\frac{d^2(\rho s)}{dt^2}$, we must put

$$\frac{\frac{d^2(\rho s)}{d\theta^2}}{\left(\frac{dt}{d\theta}\right)^2} - \frac{\frac{d(\rho s)}{d\theta}.\frac{d^2t}{d\theta^2}}{\left(\frac{dt}{d\theta}\right)^3},$$

$$\text{or } \frac{d^2(\rho s)}{d\theta^2}\left(\frac{h^2+2\int_\theta T\rho^3}{\rho^4}\right) - \frac{d(\rho s)}{d\theta}\frac{d\rho}{d\theta}\frac{2}{\rho}\left(\frac{h^2+2\int_\theta T\rho^3}{\rho^4}\right) + \frac{d(\rho s)}{d\theta}\frac{T}{\rho},$$

which changes the equation to

$$\left\{\frac{d^2(\rho s)}{d\theta^2} - \frac{2}{\rho}.\frac{d(\rho s)}{d\theta}\frac{d\rho}{d\theta}\right\}\left(\frac{h^2+2\int_\theta T\rho^3}{\rho^4}\right) + \frac{d(\rho s)}{d\theta}.\frac{T}{\rho} + S = 0.$$

$$\text{Let } \rho = \frac{1}{u};$$

$$\therefore \frac{d(\rho s)}{d\theta} = \frac{d\left(\frac{s}{u}\right)}{d\theta} = \frac{\frac{ds}{d\theta}}{u} - \frac{s\frac{du}{d\theta}}{u^2},$$

$$\text{also } -\frac{2}{\rho}\cdot\frac{d\rho}{d\theta} = \frac{2}{u}\cdot\frac{du}{d\theta};$$

$$\therefore -\frac{2}{\rho}\frac{d\rho}{d\theta}\cdot\frac{d(\rho s)}{d\theta} = \frac{2\frac{ds}{d\theta}\cdot\frac{du}{d\theta}}{u^2} - \frac{2s\left(\frac{du}{d\theta}\right)^2}{u^3}$$

$$\text{And } \frac{d^2(\rho s)}{d\theta^2} = \frac{1}{u}\frac{d^2 s}{d\theta^2} - \frac{2\frac{ds}{d\theta}\cdot\frac{du}{d\theta}}{u^2} + \frac{2s\left(\frac{du}{d\theta}\right)^2}{u^3} - \frac{s\frac{d^2u}{d\theta^2}}{u^2},$$

substituting these in the equation,

$$\left(\frac{1}{u}\cdot\frac{d^2 s}{d\theta^2} - \frac{s}{u^2}\cdot\frac{d^2u}{d\theta^2}\right)u^4\cdot\left(h^2 + 2\int_\theta \frac{T}{u^3}\right) + T\left(\frac{ds}{d\theta} - \frac{s}{u}\cdot\frac{du}{d\theta}\right) + S = 0,$$

$$\text{or } u\frac{d^2 s}{d\theta^2} - s\frac{d^2 u}{d\theta^2} + \frac{\frac{S}{u^2} + T\left(\frac{1}{u^2}\cdot\frac{ds}{d\theta} - \frac{s}{u^3}\cdot\frac{du}{d\theta}\right)}{h^2 + 2\int_\theta \frac{T}{u^3}} = 0.$$

But, by equation (d),

$$s\frac{d^2u}{d\theta^2} + su + \frac{-\frac{Ps}{u^2} + \frac{Ts}{u^3}\cdot\frac{du}{d\theta}}{h^2 + 2\int_\theta \frac{T}{u^3}} = 0.$$

Adding this to the last,

$$u\frac{d^2 s}{d\theta^2} + su + \frac{\frac{S - Ps}{u^2} + \frac{T}{u^2}\cdot\frac{ds}{d\theta}}{h^2 + 2\int_\theta \frac{T}{u^3}} = 0,$$

$$\text{or } \frac{d^2 s}{d\theta^2} + s + \frac{\frac{S - Ps}{u^3} + \frac{T}{u^3} \cdot \frac{ds}{d\theta}}{h^2 + 2\int_\theta \frac{T}{u^3}} = 0 \ldots\ldots (e).$$

34. By the solution of the two equations (d) and (e), we must express u and s in terms of θ, which will give the form and position of the orbit. The time of describing any part will then be found by integrating the equation

$$\frac{dt}{d\theta} = \frac{\rho^2}{\sqrt{h^2 + 2\int_\theta T\rho^4}} = \frac{1}{u^2\sqrt{h^2 + 2\int_\theta \frac{T}{u^3}}}.$$

35. Prop. 11. To expand the expressions for $\frac{P}{u^2}$, $\frac{T}{u^3}$, and $\frac{S - Ps}{}$

The expressions for P and T were found in Prop. 7: and

$$S - Ps = m'.r'.s.\cos(\theta - \theta')\left(\frac{1}{y'^3} - \frac{1}{y^3}\right)$$

$$\text{Now, } y^2 = \overline{m'A}|^2 + \overline{AE}|^2 \text{ (fig. 2.)}$$

$$= r'^2 + 2r'.GA.\cos\overline{\theta - \theta'} + GA^2 + GA^2.s^2.$$

Expanding as far as $\left(\frac{GA}{r'}\right)^2$,

$$\frac{1}{y^3} = \frac{1}{r'^3}\left\{1 - 3.\frac{GA}{r'}\cos\overline{\theta - \theta'} + \frac{GA^2}{r'^2}\left(\frac{9}{4} + \frac{15}{4}\cos 2\overline{\theta - \theta'}\right)\right\}.$$

$$\text{But } GA = \frac{M}{E+M}.\rho; \therefore \frac{1}{y^3}$$

$$= \frac{1}{r'^3}\left\{1 - 3\frac{M}{E+M}.\frac{\rho}{r'}.\cos\overline{\theta - \theta'} + \frac{M^2}{(E+M)^2}.\frac{\rho^2}{r'^2}\left(\frac{9}{4} + \frac{15}{4}\cos 2\overline{\theta - \theta'}\right)\right\}$$

Similarly, $\frac{1}{y'^3}$

$$=\frac{1}{r'^3}\left\{1+3\frac{E}{E+M}\cdot\frac{\rho}{r'}\cos\overline{\theta-\theta'}+\frac{E^2}{(E+M)^2}\cdot\frac{\rho^2}{r'^2}\left(\frac{9}{4}+\frac{15}{4}\cos 2\overline{\theta-\theta'}\right)\right\};$$

$$\frac{1}{y'^3}-\frac{1}{y^3}$$

$$=\frac{1}{r'^3}\left\{3\frac{\rho}{r'}\cos\overline{\theta-\theta'}+\frac{E-M}{E+M}\cdot\frac{\rho^2}{r'^2}\left(\frac{9}{4}+\frac{15}{4}\cos 2\overline{\theta-\theta'}\right)\right\}$$

Hnce, $\cos\overline{\theta-\theta}\left(\frac{1}{y'^3}-\frac{1}{y^3}\right)$

$$=\frac{1}{r'^3}\left\{\frac{3}{2}\cdot\frac{\rho}{r'}(1+\cos 2.\overline{\theta-\theta'})+\frac{E-M}{E+M}\cdot\frac{\rho^2}{r'^2}\cdot\left(\frac{33}{8}\cos\overline{\theta-\theta'}+\frac{15}{8}\cos 3.\overline{\theta-\theta'}\right)\right\},$$

$$\sin\overline{\theta-\theta'}\left(\frac{1}{y'^3}-\frac{1}{y^3}\right)$$

$$=\frac{1}{r'^3}\left\{\frac{3}{2}\cdot\frac{\rho}{r'}\sin 2.\overline{\theta-\theta'}+\frac{E-M}{E+M}\cdot\frac{\rho^2}{r'^2}\left(\frac{3}{8}\sin\overline{\theta-\theta'}+\frac{15}{8}\sin 3.\overline{\theta-\theta'}\right)\right\}.$$

And since $EG=AG\sqrt{1+s^2}=\frac{M}{E+M}\rho\sqrt{1+s^2}$,

and $mG=\frac{E}{E+M}\rho\sqrt{1+s^2}$, we find

$$\frac{1}{\sqrt{1+s^2}}\left(\frac{MG}{y'^3}+\frac{EG}{y^3}\right)=\frac{\rho}{r'^3}\left(1+3\frac{E-M}{E+M}\cdot\frac{\rho}{r'}\cos\overline{\theta-\theta'}\right),$$

neglecting powers of ρ above the square.

Substituting these values, putting $E+M=\mu$, and expanding $\frac{1}{(1+s^2)^{\frac{3}{2}}}$ as far as s^4,

$$P = \frac{\mu}{\rho^2}\left(1 - \frac{3}{2}s^2 + \frac{15}{8}s^4\right)$$

$$m'\left\{\frac{\rho}{r'^3}\left(\frac{1}{2} + \frac{3}{2}\cos 2.\overline{\theta-\theta'}\right) + \frac{E-M}{\mu}.\frac{\rho^2}{r'^4}\left(\frac{9}{8}\cos\overline{\theta-\theta'} + \frac{15}{8}\cos 3.\overline{\theta-\theta'}\right)\right\},$$

$$T = -m'\left\{\frac{3}{2}.\frac{\rho}{r'^3}.\sin 2.\overline{\theta-\theta'} + \frac{E-M}{\mu}.\frac{\rho^2}{r'^4}\left(\frac{3}{8}\sin\overline{\theta-\theta'} + \frac{15}{8}\sin 3.\overline{\theta-\theta'}\right)\right\},$$

$$S - Ps = m's.\frac{\rho}{r'^3}\left(\frac{3}{2} + \frac{3}{2}\cos 2.\overline{\theta-\theta'}\right).$$

Now, putting $\frac{1}{u}$ for ρ, and $\frac{1}{u'}$ for r', we find, at length,

$$= \mu\left(1 - \frac{3}{2}s^2 + \frac{15}{8}s^4\right)$$

$$n'\left\{\frac{u'^3}{u^3}\left(\frac{1}{2} + \frac{3}{2}\cos 2\overline{\theta-\theta'}\right) + \frac{E-M}{\mu}.\frac{u'^4}{u^4}\left(\frac{9}{8}\cos\overline{\theta-\theta'} + \frac{15}{8}\cos 3.\overline{\theta-\theta'}\right)\right\} \ldots (f),$$

$$= -m'\left\{\frac{3}{2}.\frac{u'^3}{u^4}\sin 2.\overline{\theta-\theta'} + \frac{E-M}{\mu}.\frac{u'^4}{u^5}\left(\frac{3}{8}\sin\theta-\theta' + \frac{15}{8}\sin 3.\theta-\theta'\right)\right\} \ldots (g),$$

$$\frac{S-Ps}{u^3} = m's.\frac{u'^3}{u^4}.\left(\frac{3}{2} + \frac{3}{2}\cos 2\,\overline{\theta-\theta'}\right) \ldots\ldots\ldots\ldots (h).$$

36. It appears, then, that upon substituting these values in the equation (d), it will be reduced to this form, $\frac{d^2u}{d\theta^2} + u + \Pi = 0$, Π being a complicated function of u, s, and θ. No method of directly solving such an equation is known: but we have seen in Prop. 1, that it could be solved, if Π were a function of θ only. This suggests the method of solving by successive substitution. Find a value of u in terms of θ, which is nearly the true one: substitute this value for u, in the terms of small magnitude; Π will then be a function of θ only, and the equation may be solved, and a more approximate value of u found. Substitute this for u in Π, and again solve the equation,

and a value will be found still nearer the truth. Proceed in the same manner to find the value of s.

37. But, in order to carry on this process with facility, it is necessary to establish some rule with regard to the comparative value of small quantities, so that, fixing upon some quantity as a standard, our first approximation may include its first power, and the first powers of quantities nearly as great; our second approximation may comprehend its square, and the squares of the others, and the products of any two, &c. Thus, let e be the eccentricity of the lunar orbit: e' that of the solar orbit: k the tangent of the mean inclination of the lunar orbit to the ecliptic: m the ratio of the Sun's mean motion to the Moon's mean motion. Here $e = \frac{1}{20}$ nearly; $e' = \frac{1}{60}$; $k = \frac{1}{12}$; $m = \frac{1}{13}$: taking e, then, as our standard, e', k, and m, are small quantities, not differing much in magnitude from e, and are therefore said to be small quantities of the first order. But, $\frac{\rho}{r}$ or $\frac{u'}{u}$ is little more than $\frac{1}{400}$, and therefore admits better of being compared with e^2 than with e: it is on that account considered to be a small quantity of the second order: $m^2 e$, $\frac{\rho}{r} e'$, &c. would be called of the third order; &c.

38. It is of importance to determine what is the order of the disturbing force on the Moon, compared with the force which is independent of the Sun's action. Upon examining the expressions for P, T, and S, it will be seen that the mutual attraction of the Earth and Moon is expressed by $\frac{\mu}{\rho^2}$, while the disturbing force is given by a multiple of $\frac{m'\rho}{r'^3}$. We must therefore find the order of $\frac{m'\rho}{r'^3}$, as compared with $\frac{\mu}{\rho^2}$,

or of $\frac{m'}{r'^3}$, compared with $\frac{\mu}{\rho^3}$. Now, by (15), if T be the time of a revolution of the Moon about the Earth, in a circular orbit, (the disturbing force and the ellipticity being very small) $T = \frac{2\pi \cdot \rho^{\frac{3}{2}}}{\sqrt{\mu}}$ nearly: and, if T' be the time in which the system of the Earth and Moon (supposed very small in comparison with the Sun) revolves round the Sun, $T' = \frac{2\pi \cdot r'^{\frac{3}{2}}}{\sqrt{m'}}$ nearly:

$$\text{hence } \frac{m'}{r'^3} : \frac{\mu}{\rho^3} :: \frac{1}{T'^2} : \frac{1}{T^2} :: m^2 : 1:$$

and the disturbing force is of the second order.

39. Prop. 12. To integrate the differential equations, first approximation.

We propose here to include small terms of the first order. Since the disturbing force is of the second order, we shall not take any terms arising from it. Thus, we have

$$\frac{P}{u^2} = \mu; \quad \frac{T}{u^3} = 0; \quad \frac{S - Ps}{u^3} = 0:$$

and the equations (d) and (e) become

$$\frac{d^2 u}{d\theta^2} + u - \frac{\mu}{h^2} = 0, \quad \frac{d^2 s}{d\theta^2} + s = 0.$$

The solution of the first is

$$u = \frac{\mu}{h^2} \{1 + e \cos(\theta - a)\} = a \{1 + e \cos(\theta - a)\},$$

putting $\frac{\mu}{h^2} = a$; that of the second,

$$s = k \cdot \sin(\theta - \gamma).$$

The first shews that the Moon's orbit is an ellipse; the second,

that the tan latitude $= k$. sin longitude from node, and therefore, that she apparently moves in a great circle.

40. PROP. 13. To integrate the differential equations, second approximation.

As terms of the second order are to be included, we shall here have the first terms of the disturbing force.

$$\frac{P}{u^2} \text{ therefore} = \mu\left(1 - \frac{3}{2}s^2\right) - \frac{m'u'^3}{u^3}\left(\frac{1}{2} + \frac{3}{2}\cos 2.\overline{\theta - \theta'}\right),$$

$$\frac{T}{u^3} = -\frac{3}{2}\frac{m'u'^3}{u^4}\sin 2.\overline{\theta - \theta'}\ ; \quad \frac{S - Ps}{u^3} = 0.$$

We have just found for u the expression $a\,(1 + e\cos\overline{\theta - \alpha})$; but it is evident, that in the substitution of this value, the terms containing e will be of the third order. We shall therefore for u put a: and for u' shall put a' (a' in the Sun's orbit corresponding to a in the Moon's). Also, for θ' we shall put the value which it would have, if the motions of the Sun and Moon were both uniform, that is, $m\theta - \beta$, $-\beta$ being the Sun's mean longitude, when $\theta = 0$. Thus,

$$\frac{P}{u^2} = \mu\left(1 - \frac{3k^2}{4} + \frac{3k^2}{4}\cos 2.\overline{\theta - \gamma}\right)$$

$$-\frac{m'a'^3}{a^3}\left(\frac{1}{2} + \frac{3}{2}\cos\overline{(2 - 2m)\theta + 2\beta}\right);$$

$$\frac{T}{u^3} = -\frac{3}{2}\cdot\frac{m'a'^3}{a^4}\cdot\sin\overline{(2 - 2m)\theta + 2\beta};$$

$$\frac{S - Ps}{u^3} = 0.$$

41. In this and succeeding approximations, it will be found most convenient to put the differential equations (d) and (e) into the following form;

$$\frac{d^2u}{d\theta^2}+u+\left(\frac{d^2u}{d\theta^2}+u\right)2\int_\theta\frac{T}{h^2u^3}-\frac{P}{h^2u^2}+\frac{T}{h^2u^3}\cdot\frac{du}{d\theta}=0\ldots\ldots(k),$$

$$\frac{d^2s}{d\theta^2}+s+\left(\frac{d^2s}{d\theta^2}+s\right)2\int_\theta\frac{T}{h^2u^3}+\frac{S-Ps}{h^2u^3}+\frac{T}{h^2u^3}\cdot\frac{ds}{d\theta}=0\ldots(l).$$

Now, with the assumed value of u, $\frac{d^2u}{d\theta^2}+u=a$,

$$\int_\theta\frac{T}{h^2a^3}=\frac{3}{2(2-2m)}\cdot\frac{m'a'^3}{h^2a^4}\cos\overline{(2-2m)\theta+2\beta}$$

$$=\frac{3}{4}\cdot\frac{m'a'^3}{h^2a^4}\cos\overline{(2-2m)\theta+2\beta},$$

(since the preservation of m, in the denominator, would introduce a term of the third order)

$$\therefore\left(\frac{d^2u}{d\theta^2}+u\right)2\int_\theta\frac{T}{h^2a^3}=\frac{3}{2}\cdot\frac{m'a'^3}{h^2a^3}\cos\overline{(2-2m)\theta+2\beta}.$$

Then $$\frac{-P}{h^2u^2}=-a\left(1-\frac{3k^2}{4}+\frac{3k^2}{4}\cos 2.\overline{\theta-\gamma}\right)$$

$$+\frac{m'a'^3}{h^2a^3}\left(\frac{1}{2}+\frac{3}{2}\cos\overline{(2-2m)\theta+2\beta}\right).$$

And $\frac{T}{h^2u^3}\cdot\frac{du}{d\theta}$, since $\frac{du}{d\theta}$ involves e, is of the third order.

Hence, the equation becomes

$$0=\frac{d^2u}{d\theta^2}+u-a+\frac{3k^2a}{4}+\frac{m'a'^3}{2h^2a^3}-\frac{3k^2a}{4}\cos 2.\overline{\theta-\gamma}$$

$$+2\frac{m'a'^3}{h^2a^3}\cos\overline{(2-2m)\theta+2\beta}$$

The integral of this equation gives, by (4), $u=$

$$a-\frac{3k^2a}{4}-\frac{m'a'^3}{2h^2a^3}+ae.\cos\overline{\theta-\alpha}-\frac{3k^2a}{4(2^2-1)}\cos 2.\overline{\theta-\gamma}$$

$$+\frac{3m'a'^3}{h^2a^3(\overline{2-2m}^2-1)}\cos\overline{(2-2m)\theta+2\beta}.$$

Or, taking the last term to the second order only, $u=$

$$a-\frac{3k^2a}{4}-\frac{m'a'^3}{2h^2a^3}+ae\cos\overline{\theta-\alpha}-\frac{k^2a}{4}\cos 2.\overline{\theta-\gamma}$$

$$+\frac{m'a'^3}{h^2a^3}\cos\overline{(2-2m)\theta+2\beta}.$$

To take away $\frac{m'}{h^2}$, we observe that a' and a are nearly the reciprocals of r' and ρ, which occur in (38): and, since

$$\frac{m'}{r'^3}=m^2.\frac{\mu}{\rho^3},$$

we have $m'a'^3=m^2.\mu.a^3$; $\therefore\ \frac{m'a'^3}{h^2a^3}=m^2.\frac{\mu}{h^2}=m^2a.$

Thus, the value of u becomes

$$a\left\{1-\frac{3k^2}{4}-\frac{m^2}{2}+e\cos\overline{\theta-\alpha}-\frac{k^2}{4}.\cos 2.\overline{\theta-\gamma}+m^2\cos\overline{(2-2m)\theta+2\beta}\right\}.$$

The value of s is still $k.\sin\overline{\theta-\gamma}$, all the terms of its equation being of the third order.

42. Prop. 14. To integrate the differential equations, third approximation.

Since the disturbing force is now to be taken to the third order, the value of $\frac{m'u'^3}{h^2u^3}$, or $\frac{m'a'^3}{h^2a^3}\cdot\frac{u'^3}{a'^3}\cdot\frac{a^3}{u^3}$, which occurs in $\frac{1}{h^2u^2}$, must be taken to the third order; and as $\frac{m'a'^3}{h^2a^3}$ is of the

second order, $\frac{u'^3}{a'^3}$ and $\frac{a^3}{u^3}$ must each be expanded to the first power of e' and e. Putting for u the value found in Prop. 12, namely, $a(1+e\cos\overline{\theta-\alpha})$,

$$\frac{a^3}{u^3}=1-3e\cos\overline{\theta-\alpha}.$$

We will stop a moment to consider the effect of this term.

43. In consequence of the introduction of this term, our equation will have the following form;

$$0=\frac{d^2u}{d\theta^2}+u+\&c.+A\cos\theta-\alpha+\&c.$$

Its integral therefore by (5), will contain, in the expression for u, one term of the form $-\frac{A}{2}\theta.\sin\overline{\theta-\alpha}$. The peculiarity of this term is, that while all the variable terms which have yet occurred, being sines or cosines, were periodical, and never exceeded a certain value, this term contains a factor θ, which admits of increase without limit, and the value of the term, instead of being confined within certain limits, may be of any magnitude. Our assumed expression then for u, viz.,

$$a(1+e\cos\overline{\theta-\alpha}),$$

was not an approximate one, since terms will be added to it, whose value may exceed its own; and, as the operations of the last proposition were carried on upon the supposition that the assumed value of u was near the truth, the results of these operations fall to the ground.

44. But a slight alteration in the form of our assumption, will extricate us from this difficulty. If we assume

$$u=a(1+e\cos\overline{c\theta-\alpha}),$$

and suppose c to differ very little from 1,

$$\text{then } \frac{d^2u}{d\theta^2} + u = a\,(1 + \overline{1 - c^2}\,.\,e\,.\cos\overline{c\theta - \alpha}),$$

which, as far as quantities of the first order, $= a$; and therefore the equation used in Prop. 12., is satisfied as well as it was before. And the equation of this proposition, viz.

$$0 = \frac{d^2u}{d\theta^2} + u + \&c. + A\cos\overline{c\theta - \alpha} + \&c.$$

$$\text{or } 0 = a\,(1 + \overline{1 - c^2}\,.\,e\cos\overline{c\theta - \alpha}) + \&c. + A\cos\overline{c\theta - \alpha} + \&c.$$

gives, by the comparison of coefficients of the same cosine,

$$1 - c^2 = -\frac{A}{ae},$$

and there is now no arc in any part of the expression for u.

45. Suppose, now, we substituted $a\,(1 + e\cos\overline{c\theta - \alpha})$ for u, in the expression for $\frac{P}{h^2u^2}(f)$. The second term depending on the disturbing force, is $-\frac{3}{2}\cdot\frac{m'u'^3}{h^2u^3}\cdot\cos 2\,.\,\overline{\theta - \theta'}$. Now u', being the reciprocal of the radius vector in the elliptical orbit which the Sun appears to describe about the Earth, will be expressed by $a'\,(1 + e'\cos\overline{\theta' - \zeta})$, ζ being the longitude of the Sun's perigee. And as $\theta' = m\theta - \beta$, nearly, by (40),

$$u' = a'\,(1 + e'.\cos\overline{m\theta - \beta - \zeta}), \text{ nearly;}$$

$$\therefore \frac{u'^3}{u^3} = \frac{a'^3\,(1 + e'\cos\overline{m\theta - \beta - \zeta})^3}{a^3\,(1 + e\cos\overline{c\theta - \alpha})^3}$$

$$= \frac{a'^3}{a^3}\,(1 + 3e'\cos\overline{m\theta - \beta - \zeta} - 3e\cos\overline{c\theta - \alpha}),$$

neglecting e^2, &c. in the expansion. Hence, the term in question

$$= -\frac{3}{2}\cdot\frac{m'a'^3}{h^2a^3}\cdot(1+3e'\cos\overline{m\theta-\beta-\zeta}-3e\cos\overline{c\theta-\alpha})\times\cos 2.\overline{\theta-\theta'}.$$

Now $\cos 2.\overline{\theta-\theta'}=\cos\overline{(2-2m)\theta+2\beta}$, nearly; and, consequently, the product of $\cos\overline{c\theta-\alpha}$, and $\cos 2.\overline{\theta-\theta'}$ will contain $\cos\overline{(2-2m+c)\theta+2\beta-\alpha}$, and $\cos\overline{(2-2m-c)\theta+2\beta+\alpha}$. But we have seen in (4), that upon solving the equation

$$0=\frac{d^2u}{d\theta^2}+u+\&c.+A\cos\overline{b\theta+B},$$

there will be in the expression for u, a term

$$\frac{A}{b^2-1}\cos\overline{b\theta+B}.$$

If then b differs little from 1, there will be a large term in the value of u. Now $2-2m-c$ is in this case; for c very nearly $=1$; $\therefore 2-2m-c=1-2m$, nearly; $\therefore (2-2m-c)^2-1=-4m$, nearly. And, since this term in the differential equation is of the third order, it will rise in the value of u to the second order. Our integration therefore to the second order is not correct, and we must repeat it, examining all the terms of the third order, and not rejecting those in which the coefficient of θ is nearly $=1$.

46. We may also remark, that the first term of $\frac{P}{h^2u^2}$ which results from the disturbing force, $-\frac{1}{2}\cdot\frac{m'u'^3}{h^2u^3}$, will contain $\cos^3\overline{\theta'-\zeta}$ or $\cos\overline{m\theta-\beta-\zeta}$, nearly, multiplied by a quantity of the third order. Since m is not nearly $=1$, the resulting term in the expression for u will also be of the third order. But when, after determining u, we proceed to integrate the expression

$\frac{dt}{d\theta} = \frac{1}{hu^2\sqrt{1 + 2\int\frac{T}{\theta h^2 u^3}}}$, upon expanding this fraction, there will be one term $C \cdot \cos \overline{m\theta - \beta - \zeta}$, C being a quantity of the third order, the integral of which will give, in the expression for t, a term $\frac{C}{m} \sin \overline{m\theta - \beta - \zeta}$, the coefficient of which is of the second order. Now the principal object in the lunar theory, is to find θ in terms of t, for which purpose, t must be found in terms of θ. It will be proper, then, to include in our equation all those terms of the third order, in which the coefficient of θ is small, as well as those in which it is nearly $= 1$.

47. Upon examining the equation for s, (l), (Art. 41.), it will be seen, that the same remarks apply to it, as to that which determines u. Instead of taking $s = k \sin \overline{\theta - \gamma}$, we must take $s = k \cdot \sin \overline{g\theta - \gamma}$: and must preserve, among the terms of the third order, all those in which the coefficient of θ is nearly $= 1$. We shall now integrate our equations accurately to the second order.

48. PROP. 15. To find the value of $\theta - \theta'$, $\sin 2 \cdot \overline{\theta - \theta'}$, and $\cos 2 \cdot \overline{\theta - \theta'}$ to the first order.

Since these in every place in which they occur, are multiplied by a quantity of the second order, we do not want to find their values to a higher order than the first. And to find θ' in terms of θ to this order, we must find t in terms of θ, and θ' in terms of t.

$$\text{Now } \frac{dt}{d\theta} = \frac{1}{hu^2\sqrt{1 + 2\int_\theta \frac{T}{h^2u^3}}};$$

which to the first order $= \frac{1}{hu^2}$

$$= \frac{1}{ha^2}\,\frac{1}{(1 + e\cos \overline{c\theta - \alpha})^2}$$

$$= \frac{1}{ha^2}(1 - 2e\cos \overline{c\theta - \alpha}):$$

integrating, and supposing $t=0$, when the Moon's mean longitude $=0$,

$$t = \frac{1}{ha^2}\left(\theta - \frac{2e}{c}\sin\overline{c\theta - \alpha}\right)$$

$$= \frac{1}{ha^2}\left(\theta - 2e\sin\overline{c\theta - \alpha}\right).$$

Also by (18), (observing that nt is there the Sun's mean anomaly) as the Sun's mean motion since t was $=0$, is nt, and therefore his mean longitude $= nt - \beta$ ($-\beta$ being his mean longitude, when $t=0$,) and his mean anomaly, consequently, $= nt - \beta - \zeta$, we have his true anomaly

$$= \overline{nt - \beta - \zeta} + 2e'.\sin\overline{nt - \beta - \zeta};$$

$\therefore\ \theta' =$ Sun's true longitude

$$= nt - \beta + 2e'.\sin\overline{nt - \beta - \zeta}.$$

$$\text{Now } nt - \beta = \frac{n}{ha^2}\left(\theta - 2e\sin\overline{c\theta - \alpha}\right) - \beta:$$

or, since the coefficient of θ must be m,

$$nt - \beta = m\theta - \beta,$$

(neglecting $-2me.\sin\overline{c\theta - \alpha}$, which is of the second order;)

$$\therefore\ \sin\overline{nt - \beta - \zeta} = \sin\overline{m\theta - \beta - \zeta};$$

$$\theta' = \overline{m\theta - \beta} + 2e'.\sin\overline{m\theta - \beta - \zeta},$$

$$\text{and } \overline{\theta - \theta'} = \overline{(1-m)\theta + \beta} - 2e'.\sin\overline{m\theta - \beta - \zeta}:$$

$$2.\overline{\theta - \theta'} = \overline{(2-2m)\theta + 2\beta} - 4e'.\sin\overline{m\theta - \beta - \zeta}.$$

49. We have now to find to the first order

$\sin\{\overline{(2-2m)\theta + 2\beta - p}\}$, and $\cos\{\overline{(2-2m)\theta + 2\beta - p}\}$,

p being of the first order.

$$\text{Now } \sin \overline{\{(2-2m)\theta+2\beta-p)}$$

$$= \sin \overline{(2-2m)\theta+2\beta} \,.\, \cos p - \cos \overline{(2-2m)\theta+2\beta} \,.\, \sin p.$$

But $\cos p$ differs from 1 only by a quantity of the second order, and $\sin p$ differs from p only by a quantity of the third order;

$$\therefore \sin \{\overline{(2-2m)\theta+2\beta-p}\}$$

$$= \sin \overline{(2-2m)\theta+2\beta} - p \,.\, \cos \overline{(2-2m)\theta+2\beta} :$$

which, putting for p its value $4e'. \sin \overline{m\theta-\beta-\zeta}$, gives

$$\sin 2\,.\,\overline{\theta-\theta'} = \sin \overline{(2-2m)\theta+2\beta}$$

$$- 4e'. \sin \overline{m\theta-\beta-\zeta} \,.\, \cos \overline{(2-2m)\theta+2\beta}$$

$$= \sin \overline{(2-2m)\theta+2\beta} - 2e' \sin \overline{(2-m)\theta+\beta-\zeta}$$

$$+ 2e'. \sin \overline{(2-3m)\theta+3\beta+\zeta}.$$

Similarly,

$$\cos 2\,.\,\overline{\theta-\theta'} = \cos \overline{(2-2m)\theta+2\beta} - 2e'. \cos \overline{(2-m)\theta+\beta-\zeta}$$

$$+ 2e'. \cos \overline{(2-3m)\theta+3\beta+\zeta}.$$

50. Prop. 16. To find the value of $\dfrac{P}{h^2u^2}$ to the third order.

The first part is $\dfrac{\mu}{h^2}\left(1-\dfrac{3}{2}s^2\right)$,

$$\text{or } a\left(1-\frac{3k^2}{4}+\frac{3k^2}{4}\cos \overline{2g\theta-2\gamma}\right).$$

The second part is $-\dfrac{m'u'^3}{2h^2u^3}$,

$$\text{or } -\frac{m'a'^3}{2h^2a^3} \,.\, (1+e'\cos \overline{\theta'-\zeta})^3 . (1+e\cos \overline{c\theta-\alpha})^{-3}$$

$$\frac{m'a'^3}{2h^2a^3} \times (1+3e'\cos\overline{m\theta-\beta-\zeta}-3e\cos\overline{c\theta-\alpha})$$

$$= \text{as in (41.)}, -\frac{m^2a}{2}.(1+3e'\cos\overline{m\theta-\beta-\zeta}-3e\cos\overline{c\theta-\alpha}).$$

Since the coefficient of θ in the first arc is small, and in the second, is nearly $=1$, all the terms must be preserved.

The third part is

$$-\frac{3}{2}\cdot\frac{m'u'^3}{h^2u^3}.\cos 2.\overline{\theta-\theta'},$$

$$\text{or } -\frac{3}{2}m^2a.(1+3e'\cos\overline{m\theta-\beta-\zeta}-3e\cos\overline{c\theta-\alpha})$$

$$\times\{\cos\overline{(2-2m)\theta+2\beta}-2e'\cos\overline{(2-m)\theta+\beta-\zeta}$$

$$+2e'.\cos\overline{(2-3m)\theta+3\beta+\zeta}\}.$$

Multiplying together these series, and setting down all the terms of the third order, we have

$$-\frac{3}{2}m^2a\left\{\begin{aligned}&\cos\overline{(2-2m)\theta+2\beta}+\frac{3e'}{2}\cos\overline{(2-m)\theta+\beta-\zeta}\\&+\frac{3e'}{2}\cos\overline{(2-3m)\theta+3\beta+\zeta}\\&-\frac{3e}{2}\cos\overline{(2-2m+c)\theta+2\beta-\alpha}\\&-\frac{3e}{2}\cos\overline{(2-2m-c)\theta+2\beta+\alpha}\\&-2e'\cos\overline{(2-m)\theta+\beta-\zeta}\\&+2e'\cos\overline{(2-3m)\theta+3\beta+\zeta}.\end{aligned}\right.$$

Of this the only part which must be preserved, is

$$-\frac{3}{2}m^2a\left\{\cos\overline{(2-2m)\theta+2\beta}-\frac{3e}{2}\cos\overline{(2-2m-c)\theta+2\beta+\alpha}\right\}.$$

The remaining terms of $\frac{P}{h^2u^2}$ are of the fourth order; hence, our value of $\frac{P}{h^2u^2}$ is

$$a\left(1-\frac{3k^2}{4}+\frac{3k^2}{4}\cos\overline{2g\theta-2\gamma}\right)$$

$$-\frac{m^2a}{2}\left\{1+3e'\cos\overline{m\theta-\beta-\zeta}-3e\cos\overline{c\theta-\alpha}+3\cos\overline{(2-2m)\theta+2\beta}\right.$$

$$\left.-\frac{9e}{2}\cos\overline{(2-2m-c)\theta+2\beta+\alpha}\right\}.$$

51. Prop. 17. To find the values of $\frac{T}{h^2u^3}$, $\int_\theta\frac{T}{h^2u^3}$, $\left(\frac{d^2u}{d\theta^2}+u\right)2\int_\theta\frac{T}{h^2u^3}$, and $\frac{T}{h^2u^3}\cdot\frac{du}{d\theta}$, to the third order.

The only term of $\frac{T}{h^2u^3}$ to be taken here, is

$$-\frac{3}{2}\cdot\frac{m'u'^3}{h^2u^4}\cdot\sin 2\,.\,\overline{\theta-\theta'},$$

(the others being of the fourth order), or, as in (50),

$$-\frac{3}{2}m^2\left(1+3e'\cos\overline{m\theta-\beta-\zeta}-4e\cos\overline{c\theta-\alpha}\right)\times$$

$$\left\{\sin\overline{(2-2m)\theta+2\beta}-2e'.\sin\overline{(2-m)\theta+\beta-\zeta}\right.$$

$$\left.+2e'.\sin\overline{(2-3m)\theta+3\beta+\zeta}\right\},\ \text{which}$$

$$= -\frac{3}{2}m^2 \left\{ \begin{array}{l} \sin \overline{(2-2m)\theta+2\beta} + \frac{3e'}{2}\sin \overline{(2-m)\theta+\beta-\zeta} \\ \quad - \frac{3e'}{2}\sin \overline{(2-3m)\theta+3\beta+\zeta} \\ \quad - 2e \sin \overline{(2-2m+c)\theta+2\beta-a} \\ \quad - 2e \sin \overline{(2-2m-c)\theta+2\beta+a} \\ \quad - 2e' \sin \overline{(2-m)\theta+\beta-\zeta} \\ \quad + 2e' \sin \overline{(2-3m)\theta+3\beta+\zeta}. \end{array} \right.$$

Since $\frac{T}{h^2u^3}$, and $\int_\theta \frac{T}{h^2u^3}$ are not multiplied by any circular functions, whose coefficient is not small, we shall, at this stage, reject the terms of the third order, in which the multiplier of θ differs much from 1. Thus, $\frac{T}{h^2u^3}$

$$= -\frac{3}{2}m^2 \left\{\sin \overline{(2-2m)\theta+2\beta} - 2e \sin \overline{(2-2m-c)\theta+2\beta+a}\right\}$$

52. Now, since $u = a(1 + e\cos \overline{c\theta - a})$,

$$\frac{du}{d\theta} = -cae \sin \overline{c\theta - a} = -ae \sin \overline{c\theta - a}, \text{ very nearly;}$$

$$\therefore \frac{T}{h^2u^3}\cdot\frac{du}{d\theta} = \frac{3}{4}m^2ae\left\{\cos \overline{(2-2m-c)\theta+2\beta+a} - \cos \overline{(2-2m+c)\theta+2\beta-a}\right\}.$$

The only term to be preserved, is

$$\frac{3}{4}am^2e \cdot \cos \overline{(2-2m-c)\theta+2\beta+a}.$$

53. Also $\int_\theta \frac{T}{h^2u^3} = \frac{3}{2}m^2 . \left\{\frac{1}{2-2m}\cos\overline{(2-2m)\theta+2\beta}\right.$

$$\left. - \frac{2e}{2-2m-c}\cos\overline{(2-2m-c)\theta+2\beta+a}\right\},$$

which, taking to the second order only the term that will not be increased by integration, and taking the other only to the third order, is

$$\frac{3}{2}m^2 \{\tfrac{1}{2}\cos\overline{(2-2m)\theta+2\beta} - 2e . \cos\overline{(2-2m-c)\theta+2\beta+a}\}.$$

And $\frac{d^2u}{d\theta^2} + u$, to the first order, is a;

$$\therefore 2\left(\frac{d^2u}{d\theta^2}+u\right)\int_\theta \frac{T}{h^2u^3} = 3m^2a\,\{\tfrac{1}{2}\cos\overline{(2-2m)\theta+2\beta}$$

$$- 2e\cos\overline{(2-2m-c)\theta+\beta+a}\}.$$

54. Prop. 18. To form the differential equation for u.

Collecting the terms, and substituting them in the equation of (41), we have

$$\frac{d^2u}{d\theta^2}+u$$

$$= \frac{d^2u}{d\theta^2}+u$$

$$- \left(\frac{d^2u}{d\theta^2}+u\right) 2\int_\theta \frac{T}{h^2u^3}$$

$$= 3m^2a\,\{\tfrac{1}{2}\cos\overline{(2-2m)\theta+2\beta}$$

$$- 2e\cos\overline{(2-2m-c)\theta+2\beta+a}\} \text{ (Prop. 17.)}$$

$$- \frac{P}{h^2u^2}$$

$$= -a\left(1-\frac{3k^2}{4}+\frac{3k^2}{4}\cos\overline{2g\theta-2\gamma}\right)+\frac{m^2a}{2}$$

$$+\frac{m^2a}{2}(3e'.\cos\overline{m\theta-\beta-\zeta}-3e\cos\overline{c\theta-a})$$

$$+\frac{m^2a}{2}\{3\cos\overline{(2-2m)\theta+2\beta}-\frac{9e}{2}\cos\overline{(2-2m-c)\theta+2\beta+a}\}$$

(Prop. 16.)

$$+\frac{T}{h^2u^3}\cdot\frac{du}{d\theta}$$

$$=+\frac{-}{4}m^2ae.\cos\overline{(2-2m-c)\theta+2\beta+a}. \quad \text{(Prop. 17.)}$$

Taking the sum, by (41),

$$0=\frac{d^2u}{d\theta^2}+u-a\left(1-\frac{3k^2}{4}-\frac{m^2}{2}\right)$$

$$3m^2ae\ .\cos\overline{c\theta-a} \qquad 3k^2a\ \cos\overline{2g\theta-2\gamma}$$

$$+3m^2a.\cos\overline{(2-2m)\theta+2\beta}-\frac{15}{2}.m^2ae\cos\overline{(2-2m-c)\theta+2\beta+a}$$

$$+\frac{3}{2}m^2ae'.\cos\overline{m\theta-\beta-\zeta}.$$

55. Prop. 19. To integrate accurately to the second order the differential equation for u.

$$\text{Assume } u=a\{1-\frac{3k^2}{4}-\frac{m^2}{2}+e.\cos\overline{c\theta-a}$$

$$+A\cos\overline{2g\theta-2\gamma}+B\cos\overline{(2-2m)\theta+2\beta}$$

$$+C\cos\overline{(2-2m-c)\theta+2\beta+a}+D\cos\overline{m\theta-\beta-\zeta}\},$$

according to the direction in (6).

Substituting this value in the differential equation, and making $=0$ the coefficient of each cosine,

$$ae(1-c^2)-\frac{3m^2ae}{2}=0;$$

$$c^2=1-\frac{3m^2}{2},\ c=1-\frac{3m^2}{4},\ \text{nearly.}$$

$$aA(1-4g^2)-\frac{3k^2a}{4}=0;$$

$$A=\frac{3k^2}{4(1-4g^2)}=-\frac{k^2}{4},\ \text{nearly, since } g \text{ nearly}=1.$$

$$aB(1-\overline{2-2m}|)+3m^2a=0;$$

$$B=\frac{3m^2}{\overline{2-2m}|^2-1}=m^2,\ \text{nearly.}$$

$$aC(1-\overline{2-2m-c}|^2)-\frac{15}{2}m^2ae=0;$$

$$C=\frac{15}{2}\cdot\frac{\quad}{1-\overline{2-2m-c}|^2}-\frac{15}{2}\cdot\frac{m^2e}{1-\overline{1-2m}|^2},\ \text{nearly,}$$

$$(\text{since } c \text{ nearly}=1)=\frac{15}{8}me,\ \text{nearly.}$$

$$aD(1-m^2)+\frac{3}{2}m^2ae'=0;$$

$$\therefore\ D=\frac{3}{2}\cdot\frac{m^2e'}{1-m^2}=-\frac{3}{2}m^2e',\ \text{nearly.}$$

And u therefore $=a\left\{1-\frac{3k^2}{4}-\frac{m^2}{2}+e\cos\overline{c\theta-\alpha}\right.$

$$-\frac{\kappa}{4}\cos\overline{2g\theta-2\gamma}+m^2\cos\overline{(2-2m)\theta+2\beta}$$

$$\left.+\frac{15}{8}me\cos\overline{(2-2m-c)\theta+2\beta+\alpha}-\frac{3}{2}m^2e'.\cos\overline{m\theta-\beta-\zeta}\right\}.$$

56. PROP. 20. To form the differential equation for s.

s, as we have observed, will be approximately represented by $k \sin \overline{g\theta - \gamma}$, where g differs little from 1; the difference, which is caused entirely by the disturbing force, being of the second order. Hence,

$$\frac{d^2 s}{d\theta^2} + s, \text{ or } k(1-g^2) \sin \overline{g\theta - \gamma}$$

will be small, of the third order: consequently, the term

$$\left(\frac{d^2 s}{d\theta^2} + s\right) 2 \int_\theta \frac{T}{h^2 u^3}:$$

in equation (l) of Art. 41, will be of the fifth order, and is not to be considered.

$$\frac{S - Ps}{h^2 u^3}$$

$$= \frac{m' a'^3}{h^2 a^4} \frac{(1 + e \cos \overline{m\theta - \beta - \zeta})^3}{(1 + e \cos \overline{c\theta - \alpha})^4} . k \sin \overline{g\theta - \gamma} . \left(\frac{3}{2} + \frac{3}{2} \cos 2.\overline{\theta - \theta'}\right).$$

Now $\frac{m' a'^3}{h^2 a^4} k$ or $m^2 k$, form a product of the third order; hence in the quantities which multiply them, all small terms are to be rejected;

$$\therefore \frac{S - Ps}{h^2 u^3} = m^2 k . \sin \overline{g\theta - \gamma} \left(\frac{3}{2} + \frac{3}{2} \cos \overline{(2 - 2m)\theta + 2\beta}\right)$$

$$= m^2 k \left(\frac{3}{2} \sin \overline{g\theta - \gamma} + \frac{3}{4} \sin \overline{(2 - 2m + g)\theta + 2\beta - \gamma}\right.$$

$$\left. - \frac{3}{4} \sin \overline{(2 - 2m - g)\theta + 2\beta + \gamma}\right).$$

The only terms to be preserved are

$$m^2 k \left(\frac{3}{2} \sin \overline{g\theta - \gamma} - \frac{3}{4} \sin \overline{(2 - 2m - g)\theta + 2\beta + \gamma}\right).$$

$$\frac{ds}{d\theta} = kg . \cos \overline{g\theta - \gamma} = k . \cos \overline{g\theta - \gamma} \text{ nearly,}$$

which is of the first order; taking therefore the first term only of the expression for $\frac{T}{h^2u^3}$, that is, (51)

$$-\frac{3}{2}m^2 . \sin \overline{(2-2m)\,\theta+2\beta},$$

we have

$$\frac{T}{h^2u^3}\cdot\frac{ds}{d\theta}=-\frac{3}{2}m^2k\,.\cos\overline{g\theta-\gamma}\,.\sin\overline{(2-2m)\,\theta+2\beta}$$

$$=-\frac{3}{4}m^2k\,\{\sin\overline{(2-2m+g)\,\theta+2\beta-\gamma}+\sin\overline{(2-2m-g)\,\theta+2\beta+\gamma}\}.$$

The only term to be preserved is

$$-\frac{3}{4}m^2k\,.\sin\overline{(2-2m-g)\,\theta+2\beta+\gamma}.$$

Collecting these parts, the equation of (41) becomes

$$0=\frac{d^2s}{d\theta^2}+s+\frac{3}{2}m^2k\,.\sin\overline{g\theta-\gamma}-\frac{3}{2}m^2k\sin\overline{(2-2m-g)\,\theta+2\beta+\gamma}.$$

57. Prop. 21. To integrate the differential equation for s.

Assume $s=k\,(\sin\overline{g\theta-\gamma}+A\sin\overline{(2-2m-g)\,\theta+2\beta+\gamma})$,

and substitute in the equation above: then, making $=0$ the coefficient of each sine,

$$k\,(1-g^2)+\frac{3}{2}m^2k=0;\;\therefore\; g^2=1+\frac{3}{2}m^2;$$

$$kA\,(1-\overline{2-2m-g}|^2)-\frac{3}{2}m^2k=0;$$

$$\therefore\; A=\frac{3}{2}\cdot\frac{m^2}{1-(2-2m-g)^2}$$

$$=\frac{3}{2}\cdot\frac{m^2}{1-(1-2m)^2}\text{ nearly}=\frac{3}{2}\cdot\frac{m^2}{4m}\text{ nearly}=\frac{3m}{8};$$

$$\therefore\ s=k\left(\sin\overline{g\theta-\gamma}+\frac{3m}{8}.\sin\overline{(2-2m-g)\theta+2\beta+\gamma}\right),$$

to the second order.

58. PROP. 23. To find t in terms of θ to the second order.

We must expand $\dfrac{dt}{d\theta}$ or $\dfrac{1}{hu^2\sqrt{1+2\int_\theta \frac{T}{h^2u^3}}}$ to the second order, preserving those terms of the third order in which the coefficient of θ is small. Putting then for u the value found in Prop. 19, and for $\int_\theta \frac{T}{h^2u^3}$ the first term of its value in Prop. 17,

$$\frac{dt}{d\theta}=\frac{1}{ha^2}\left\{1+\frac{3k^2}{2}+m^2-2e\cos\overline{c\theta-a}\right.$$

$$+\frac{3e^2}{2}+\frac{3e^2}{2}\cos\overline{2c\theta-2a}+\frac{k^2}{2}\cos\overline{2g\theta-2\gamma}$$

$$-2m^2\cos\overline{(2-2m)\theta+2\beta}-\frac{15}{4}me\cos\overline{(2-2m-c)\theta+2\beta+a}$$

$$\left.+3m^2e'\cos\overline{m\theta-\beta-\zeta}-\frac{3}{4}m^2\cos\overline{(2-2m)\theta+2\beta}\right\}.$$

Integrating this, and taking the coefficients to the second order only,

$$t=\frac{1}{ha^2}\left\{\left(1+\frac{3k^2}{2}+m^2+\frac{3e^2}{2}\right)\theta-2e\sin\overline{c\theta-a}\right.$$

$$+\frac{3e^2}{4}\sin\overline{2c\theta-2a}+\frac{k^2}{4}\sin\overline{2g\theta-2\gamma}$$

$$-\frac{11}{8}m^2 \sin (2-2m)\,\theta + 2\beta$$

$$-\frac{15}{4}me\,.\sin \overline{(2-2m-c)\,\theta+2\beta+\alpha} + 3me' \sin \overline{m\theta-\beta-\zeta}\Big\}$$

Let $\dfrac{ha^2}{1+\dfrac{3k^2}{2}+m^2+\dfrac{3e^2}{2}} = p$: then,

$$pt = \theta - 2e \sin c\theta - \alpha + \frac{3e^2}{4} \sin 2c\theta - 2\alpha$$

$$+\frac{k^2}{4} \sin \overline{2g\theta - 2\gamma} - \frac{11}{8} m^2 . \sin \overline{(2-2m)\,\theta + 2\beta}$$

$$-\frac{15}{4} me\,.\sin \overline{(2-2m-c)\,\theta+2\beta+\alpha} + 3me'.\sin \overline{m\theta-\beta-\zeta}.$$

59. Prop. 24. To find θ in terms of t to the second order.

This must be done by Lagrange's theorem. Applying it, we find

$$\theta = pt + 2e\,.\sin \overline{cpt - \alpha} + \frac{5e^2}{4} \sin 2cpt - 2\alpha$$

$$-\frac{k^2}{4} \sin 2gpt - 2\gamma + \frac{11}{8} m^2 . \sin (2-2m)\,pt + 2\beta$$

$$+\frac{15}{4} me\,.\sin (2-2m-c)\,pt+2\beta+\alpha - 3me'.\sin \overline{mpt-\beta-\zeta}.$$

60. Prop. 25. To find an expression for the Moon's parallax.

The Moon's parallax varies inversely as her distance, that is, it $\propto \frac{1}{r}$,

$$\text{or } \frac{1}{\rho(1+s^2)^{\frac{1}{2}}}, \text{ or } \frac{u}{(1+s^2)^{\frac{1}{2}}}, \text{ or } u\left(1-\frac{s^2}{2}\right),$$

$$\text{or } u\left(1-\frac{k^2}{4}+\frac{k^2}{4}\cos\overline{2g\theta-2\gamma}\right).$$

Taking for u the expression found in Prop. 19, parallax

$$\propto a\left(1-k^2-\frac{m^2}{2}+e\cos\overline{c\theta-\alpha}+m^2\cos\overline{(2-2m)\theta+2\beta}\right.$$

$$\left.+\frac{15}{8}me.\cos\overline{(2-2m-c)\theta+2\beta+\alpha}\right):$$

or, if P be the mean parallax, that is, that part of the expression independent of cosines, the parallax

$$=P\left(1+e\cos\overline{c\theta-\alpha}+m^2.\cos\overline{(2-2m)\theta+2\beta}\right.$$

$$\left.+\frac{15}{8}me.\cos\overline{(2-2m-c)\theta+2\beta+\alpha}\right).$$

61. Prop. 26. To explain the effect of the different terms in these expressions.

The first and greatest inequality of parallax is

$$e\cos\overline{c\left(\theta-\frac{\alpha}{c}\right)}.$$

This, though similar to the inequality which would exist in an elliptic orbit, is not exactly the same, but it would be the same if it depended on the angle $\theta-\frac{\alpha}{c}$ instead of $c\left(\theta-\frac{\alpha}{c}\right)$. Let then EA, fig. 4, be the Moon's least distance: EM any other distance: $\angle AEM=\theta-\frac{\alpha}{c}$: let AmB be an ellipse, whose latus rectum is $\frac{1}{a}$, and eccentricity e: take $\angle AEm=c\times\angle AEM$ (c being <1): then, EM will $=Em$. For

$$\frac{1}{Em} = a\,(1 + e\cos AEm) = a\,(1 + e\cos\overline{c\theta - \alpha}) = \frac{1}{EM}.$$

If now an ellipse aMb be described similar and equal to AmB, whose major axis ab is inclined to AB, at an angle equal to mEM, or $(1-c)\left(\theta - \frac{\alpha}{c}\right)$, M will evidently be found in its circumference, and the arc aM will be $= Am$. Hence, the motion of M may be represented by supposing it to move in an ellipse, and supposing that ellipse to revolve in the same direction with an angular velocity, which is to the whole angular velocity of M as $1-c : 1$. The perigee of the Moon's orbit therefore is not fixed, but (while we neglect the other terms of the parallax) moves almost uniformly in the direction of the Moon's motion. (*Newton*, Prop. 66. Cor. 7.)

62. For the explanation of the next term,

$$m^2 . \cos\overline{(2-2m)\,\theta + 2\beta},$$

we observe that $\overline{(1-m)\,\theta + \beta}$ (Prop. 15.) is nearly the difference of longitude of the Sun and Moon, and consequently,

$$\cos\overline{(2-2m)\,\theta + 2\beta}$$

is greatest, when the Moon is in syzygies, and least (or has its greatest negative value), when she is in quadratures; and between these situations, it has all the intermediate values. The Moon's parallax therefore, setting aside the elliptic inequality, is greatest in syzygies, and least in quadratures, and therefore her distance is least in syzygies, and greatest in quadratures. (*Newton*, Prop. 66. Cor. 5.)

63. The next term,

$$\frac{15}{8}\,mc\cos\overline{(2-2m-c)\,\theta + 2\beta + \alpha},$$

shews its influence by the alteration of the eccentricity of the Moon's orbit. To prove this, let us suppose that when $\theta = 0$,

the Moon, and the axis of the Moon's orbit, were in syzygies: then, (considering only the elliptic inequality, and the present one) parallax

$$=P\left(1+e\cos c\theta+\frac{15}{8}me\cos\overline{(2-2m-c)\theta}\right):$$

and, since c and $2-2m-c$ each nearly $=1$, the Moon's parallax, for one revolution, will be nearly represented by the expression

$$P\left(1+\overline{e+\frac{15}{8}me}.\cos\theta\right),$$

which is the same as in an orbit, whose eccentricity

$$=e\left(1+\frac{15}{8}m\right).$$

Again, suppose that when $\theta=0$, the Moon is at her apse, and the Sun's longitude -90^0, or $\beta=90^0$: then, when the Moon's longitude $=\theta$, her parallax

$$=P\left(1+e\cos c\theta+\frac{15}{8}me\cos\overline{(2-2m-c)\theta+180^0}\right)$$

which, for a single revolution, is nearly represented by

$$P\left(1+e\cos\theta+\frac{15}{8}me\cos\overline{\theta+180^0}\right),$$

$$\text{or }P\left(1+\overline{e-\frac{15}{8}me}.\cos\theta\right):$$

and here the apparent eccentricity is $e\left(1-\frac{15}{8}m\right)$. The eccentricity therefore is increased by this term, when the axis of the Moon's orbit is in syzygies, and diminished when it is in quadratures: its effect in intermediate situations, we shall consider presently. (*Newton*, Prop. 66. Cor. 9.)

64. Corresponding to these inequalities of parallax, there are, in the expression for the Moon's longitude, the terms

$$2e \,.\, \sin \overline{cpt - \alpha} + \frac{5e^2}{4} \sin \overline{2cpt - 2\alpha},$$

$$\frac{11}{8} m^2 \sin \overline{(2 - 2m)pt + 2\beta},$$

and $\frac{15}{4} me \,.\, \sin \overline{(2 - 2m - c)pt + 2\beta + \alpha}$ (Prop. 24.).

The first two depend on the eccentricity e and the mean distance from the perigee $cpt - \alpha$: their sum constitutes the *elliptic inequality* in longitude. The next term, which is called the *variation*, is proportional to $\sin 2 \,.\, (pt - mpt - \beta)$, or sin 2 (Moon's mean longitude − Sun's mean longitude). It is therefore $= 0$ when the Moon is in conjunction; it is greatest, when the difference of longitudes $= 45^0$; it is again $= 0$, when the difference $= 90^0$, or the Moon is at quadrature: it has its greatest negative value when that difference $= 135^0$; and is again $= 0$, when the difference of longitudes $= 180^0$, or the Moon is in opposition. The Moon's true place therefore is before the mean place from syzygy to quadrature, and behind it, from quadrature to syzygy. (*Newton*, Lib. III. Prop. 29.) The last term, depending on $\sin \overline{(2 - 2m - c)pt + 2\beta + \alpha}$, is called the *evection*: it appears to increase the elliptic inequality, when the axis of the Moon's orbit is in syzygies, and to diminish it, when that axis is in quadratures: the reasoning of the last article applies to it in every respect.

There are, besides (Prop. 24.) these terms,

$$-\frac{k^2}{4} \sin \overline{2gpt - 2g}, \text{ and } -3me' \sin \overline{mpt - \beta - \zeta}.$$

The former of these depends upon the Moon's distance from the mean place of her node, and is nearly the difference between her longitude, measured on her orbit, and her longitude,

measured on the ecliptic: it is called the *reduction.* The latter depends on the Sun's mean anomaly: it appears, that while the Sun (apparently) goes from perigee to apogee, the Moon's true place is behind her mean place: while the Sun goes from apogee to perigee, the Moon's true place is before her mean place. (*Newton*, Prop. 66. Cor. 6.) This is called the *annual equation.* The alteration in the parallax, from this cause, is very small, being of the third order.

65. In respect of magnitude, the evection is far the most important of the inequalities, which are produced by the disturbing force of the Sun. And its effect on the position and eccentricity of the Moon's orbit is so remarkable, that we shall here consider it a little more generally than in Art. 63.

66. PROP. 27. To determine the change in the position of the axis, and in the eccentricity of the Moon's orbit, produced by the evection.

The elliptic inequality and evection, are together represented by

$$\left(\cos \overline{c\theta - a} + \frac{15}{8} m \cos \overline{(2-2m-c)\theta + 2\beta + a}\right),$$

where a = longitude of perigee, if there were no evection, $-\beta$ = longitude of Sun, when $\theta = 0$. During a part of one revolution, we may, without great error, suppose the perigee and the Sun to be stationary: then, for $\overline{c\theta - a}$, we must put $\overline{\theta - a}$, and for $\overline{(2-2m)\theta + 2\beta - c\theta - a}$, or twice the distance of the Sun and Moon —the Moon's anomaly, we must put

$$2\theta + 2\beta - \theta - a = \theta - a + 2\beta + a.$$

And the united inequalities

$$= e\left(\cos \overline{\theta - a} + \frac{15}{8} m . \cos \overline{(\theta - a) + 2(\beta + a)}\right)$$

$$= e\left(1 + \frac{15}{8} m . \cos 2\overline{\beta + a}\right) . \cos \overline{\theta - a}$$

$$-e\,.\,\frac{15}{8}\,m\,.\,\sin 2\,.\,\overline{\beta+a}\,.\,\sin\overline{\theta-a}.$$

This may be put under the form $E\cos(\theta-a+\delta)$, if

$$E\cos\delta=e\left(1+\frac{15}{8}\,m\cos 2\,.\,\overline{\beta+a}\right),$$

$$E\sin\delta=e\,\frac{15}{8}\,m\,.\,\sin 2\,\overline{\beta+a}.$$

From these equations,

$$\delta \text{ or } \tan\delta=\frac{15}{8}\,m\,.\sin 2\,.\,\overline{\beta+a},\text{ nearly,}$$

$$E=e\left(1+\frac{15}{8}\,m\cos 2\,.\,\overline{\beta+a}\right).$$

And the united inequalities are represented by

$$e\left(1+\frac{15}{8}\,m\cos 2\,.\,\overline{\beta+a}\right)\cos\left(\overline{\theta-a-\frac{15}{8}\,m\sin 2\,.\,\overline{\beta+a}}\right).$$

This is the same as the expression for the elliptic inequality in an orbit whose eccentricity

$$=e\left(1+\frac{15}{8}\,m\cos 2\,.\,\overline{\beta+a}\right),$$

and in which the longitude of the perigee

$$=a-\frac{15}{8}\,m\,.\,\sin 2\,.\,\overline{\beta+a}.$$

Hence, to find the Moon's place, when we have found the longitude of the perigee, on the supposition of its uniform progression, we must subtract from that longitude

$$\frac{15}{8}\,m\,.\,\sin 2\,(\text{long. perigee}-\text{long. Sun}),$$

and apply the equation due to an elliptic orbit, whose eccentricity

$$=1+\frac{15}{8}m\,.\cos 2\,(\text{long. perigee}-\text{long. Sun}).$$

(*Newton*, Lib. III. *Scholium to the Lunar Theory*.)

67. PROP. 28. To explain the effect of the terms in the expression for s.

The first of these is $k\,.\sin g\,\overline{\left(\theta-\frac{\gamma}{g}\right)}$. If this depended on $\theta-\frac{\gamma}{g}$, it would shew that the Moon moved in a plane. But, as it depends on $g\theta-\gamma$, the Moon's motion in latitude may be represented by supposing her to move in a plane, the tangent of whose inclination to the ecliptic is k, and, supposing this plane to move with a retrograde motion, which is to the whole motion of the Moon as $g-1:1$, and which therefore is nearly uniform. This is exactly analogous to the motion of the perigee in (61), with the single difference, that c being <1, and $g>1$, the motion in one case is direct, and in the other, retrograde.

68. The second term

$$k\,.\frac{3m}{8}\,.\sin\overline{(2-2m-g)\,\theta+2\beta+\gamma},$$

has precisely the same relation to the first, which the evection has to the elliptic inequality; and the alteration which it produces in the place of the node and the inclination of the orbit, may be found in the same manner. Thus, γ is the longitude of the node, if the second term did not exist, and $-\beta$ the longitude of the Sun, when $\theta=0$. Now, during the description of a portion only of the orbit, we may, without material error, suppose in our expressions, that the Sun and node are stationary: then, for $g\theta-\gamma$, the Moon's distance from the

node, we must put $\theta-\gamma$; for $(2-2m)\theta+2\beta$, double the excess of the Moon's longitude above the Sun's, we must put $2\theta+2\beta$, and for $(2-2m-g)\theta+2\beta+\gamma$, we must put

$$2\theta+2\beta-\theta+\gamma, \quad \text{or } \theta-\gamma+2.\beta+\gamma.$$

Hence, $s=k\left(\sin\overline{\theta-\gamma}+\frac{3m}{8}\sin\overline{\theta-\gamma+2(\beta+\gamma)}\right)$

$$=k\left(1+\frac{3m}{8}\cos 2.\overline{\beta+\gamma}.\sin\overline{\theta-\gamma}+\frac{3m}{8}\sin 2\,\overline{\beta+\gamma}.\cos\overline{\theta-\gamma}\right).$$

This may be put under the form $K\sin(\theta-\gamma+\kappa)$,

$$\text{if } K\sin\kappa=\frac{3m}{8}\sin 2\,\overline{\beta+\gamma},$$

$$K\cos\kappa=1+\frac{3m}{8}\cos 2.\overline{\beta+\gamma}$$

These equations give

$$\kappa \text{ or } \tan\kappa=\frac{3m}{8}\sin 2.\overline{\beta+\gamma},$$

$$K=1+\frac{3m}{8}\cos 2.\overline{\beta+\gamma},$$

and $s=k\left(1+\frac{3m}{8}\cos 2.\overline{\beta+\gamma}\right).\sin\left(\theta-\gamma-\frac{3m}{8}\sin 2.\overline{\beta+\gamma}\right)$.

This is the same expression that we should have had, if the longitude of the node were

$$\gamma-\frac{3m}{8}\sin 2.\overline{\beta+\gamma},$$

and the tangent of the inclination of the orbit

$$k\left(1+\frac{3m}{8}\cos 2.\overline{\beta+\gamma}\right).$$

When therefore the longitude of the node is found on the supposition of its uniform retrogradation, we must subtract from it

$$\frac{3m}{8} \cdot \sin 2\,(\text{long. node} - \text{long. Sun})$$

and, taking that for the true longitude of the node, we must suppose the tangent of inclination

$$= k\left(1 + \frac{3m}{8}\cos 2\,(\text{long. node} - \text{long. Sun})\right).$$

(*Newton*, Lib. III. Prop. 33. and 35.)

69. From this expression it is evident that the inclination of the orbit is greatest when $\beta + \gamma = 0$ or 180^0, that is, when the line of nodes is in syzygies; and least, when $\beta + \gamma = 90$ or 270, that is, when the line of nodes is in quadratures. (*Newton*, Prop. 66. Cor. 10.)

70. In the same manner in which we have approximated to the value of u and θ to the second order, we might go on to the third and higher orders. For the third order, it would be necessary to examine the terms of the equation to the fourth order, and thus the last terms, in the expressions for $\frac{P}{h^2 u^2}$, &c., in Art. 35, would be employed. As the method of conducting all these approximations must be the same, we shall here mention the several steps.

(1) From the last approximate value, find t in terms of θ.

(2) Since θ' is, by the elliptic theory, found in terms of t, it can be expressed in terms of θ, and $\theta - \theta'$, $2(\theta - \theta')$, &c. can be expressed. u' also, which is known in terms of θ', can be expressed in terms of θ.

(3) Find expressions for $\sin 2(\theta - \theta')$, $\cos 2(\theta - \theta')$, &c. to as many orders as may be necessary.

(4) Substitute these values, and the last approximate value of u, in the expressions for $\frac{P}{h^2u^2}$, &c.

(5) When these are substituted in the equation, integrate it, as for the second order.

(6) Proceed in the same way in every respect, for the determination of s.

71. In carrying the approximation to higher orders, it frequently happens, that a term will rise, by integration, two orders. This renders the operations very troublesome, and particular methods are sometimes necessary: but we cannot stop to explain them here.

72. We shall here mention some of the most interesting results of the next approximation. (1) The last terms in the expressions for $\frac{P}{h^2u^2}$, &c. introduce into the equation for u multiples of

$$m^2 . \frac{a'}{a} . \frac{E-M}{\mu} . \cos (1-m)\theta+\beta,$$

which, upon integration, rise to the third order; and the corresponding inequality in longitude is of the third order. The comparison of this observed inequality, with its computed value, gives us the means of determining $\frac{a'}{a}$ $\left(\frac{M}{E}\right.$ being known pretty exactly, and therefore, $\frac{E-M}{\mu}$ or $\frac{E-M}{E+M}$ being known$\left.\right)$, that is, the ratio of the Sun's parallax to the Moon's. The latter of these is very well known from observation: hence, the former can be found. Its quantity thus determined agrees very exactly with that determined by transits of Venus.

(2) The value of c, found by the second approximation, is

$1-\frac{3m^2}{4}$, which gives, for the progression of the perigee in a revolution of the Moon, $\frac{3m^2}{4}\times 2\pi = 1^0.30'$. This is about half its true quantity, which is $3^0.2'.22''$. By continuing the approximation, we find that c is expressed by a slowly converging series, of which the first terms are

$$1-\frac{3}{4}m^2-\frac{225}{32}m^3.$$

The series for g, on the contrary, converges fast; it is

$$1+\frac{3}{4}m^2-\frac{9}{32}m^3+\&c.$$

Hence, the ratio of the motion of the perigee to the motion of the node for the Moon, is much greater than for one of Jupiter's satellites, where m is extremely small. This is alluded to by *Newton*, Lib. III. Prop. 23.

73. Upon continuing the approximations, it appears that p, the coefficient of t in the Moon's mean longitude, depends upon e', and consequently, an alteration in e' produces an alteration in the Moon's mean motion. Now e', the eccentricity of the Sun's or Earth's orbit, is slowly diminishing from the attraction of the other planets; and this causes an increase in the Moon's mean motion. It is remarkable, that the indirect effect on the Moon is much greater than the direct effect on the Earth.

74. The coefficients of inequalities of a high order can never be calculated accurately from theory. The forms can be found, and the coefficients can then be determined from observations. Thus, the most important inequality lately discovered in the motion of the Moon, was detected by Laplace, who suggested, that in the expression for $\frac{T}{h^2u^3}$, there must be

terms of the form $\sin\overline{(3-2g-c)\theta+a+2\gamma-3\zeta}$. These could result only from the multiplication of sines or cosines of these arcs; $\overline{(3-3m)\theta+3\beta}$, $\overline{3m\theta-3\beta-3\zeta}$, $\overline{2g\theta-2\gamma}$, $\overline{c\theta-a}$. The first would have for coefficient some multiple of $m^2.\frac{a'}{a}$, which would therefore be of the fourth order: the second, some multiple of e'^3, which is of the third order: the third a multiple of k^2, of the second order: and the four th multiple of e, of the first order. Hence, the coefficient of

$$\sin\overline{(3-2g-c)\,\theta+a+2\gamma-3\zeta}$$

would be of the tenth order. But, in the expression for

$$t=\int\theta\,\frac{1}{hu^2\sqrt{1+2\int_\theta\frac{T}{h^2u^3}}},$$

this would be twice integrated, and its coefficient would therefore be divided by $(3-2g-c)^2$. Now, by continued approximations, it is found that $c=0{,}991548$, $g=1{,}004022$; therefore $3-2g-c=0{,}000407$, and the divisor is $(0{,}000407)^2$. By the exceeding smallness of this divisor, the term is so much increased as to become sensible. But the calculation of the coefficient from theory is quite impracticable: this has been found by observation to be 15″.

75. In the preceding investigations, we have supposed the Sun's perigee, or the Earth's perihelion, to be stationary. It has in reality a slow progressive motion, which will be represented by putting in all the expressions $c'\theta'-\zeta$ for $\theta'-\zeta$, where $c'=0{,}999990779$.

FIGURE OF THE EARTH.

PRELIMINARY PROPOSITIONS.

1. PROP. 1. THE sections of two similar and concentric and similarly situated spheroids, made by the same plane, are similar and concentric ellipses, similarly situated.

Let BK, CG, fig. 1, be the sections, which will be ellipses (Hustler's *Conic Sections*, p. 62.). Join W, the centre of the spheroid, with N, the centre of the ellipse BK, and let ABE, CD be sections by any plane through NW: these will evidently be similar ellipses, whose common centre is W. And since BK is bisected in N, BN is an ordinate to the semi-diameter AW, and is therefore parallel to the tangent at A, and therefore to that at H (as the ellipses are similar); hence, CN is an ordinate to WH, in the smaller ellipse. Let WD and WE be the semi-conjugate diameters: then,

$$NC^2 = \frac{WD^2}{WH^2} \cdot (WH^2 - WN^2); \quad NB^2 = \frac{WE^2}{WA^2} \cdot (WA^2 - WN^2);$$

and, as the ellipses DCH, ECA, are similar,

$$\frac{WD^2}{WH^2} = \frac{WE^2}{WA^2}; \quad \therefore \frac{NC^2}{NB^2} = \frac{WH^2 - WN^2}{WA^2 - WN^2},$$

a constant value for the same sections; hence, the sections are similar, and similarly situated, with respect to N: but N is the centre of BFK; therefore it is also the centre of CG.

2. Prop. 2. Let AGB, CSK, (fig. 2.) be two ellipses, concentric, similar, and similarly situated; through C, the extremity of the axis of the smaller ellipse, draw ECO perpendicular to that axis, meeting the larger ellipse in E and O; draw EF parallel to the same axis; then, if the angles GEF, FEH, KCD, DCL, be all equal, $EG+GH$ will be equal to $CK+CL$.

3. For draw the diameter $QTNV$ bisecting EG in R; and draw OP parallel to EG.

Since EG is bisected by QV, EG is an ordinate to that diameter; so also is OP, which is parallel to EG. And because the two ellipses are similar, the tangent at Q is parallel to that at S; hence CK being parallel to EG, or parallel to the tangent at Q, is also parallel to the tangent at S; it is therefore an ordinate to the diameter SN, and is bisected by it in T. Now, because OP is parallel to EG, and EF perpendicular to EO, the angle COP is the complement of GEF; and CEH is also the complement of HEF; but $GEF=HEF$; $\therefore$ CEH is equal to COP. And the point O has the same situation in the semi-ellipse AOB, which E has in AEB. Hence, $OP=EH$. But as $CE=CO$, and ER, CT, OV, are parallel, if through T the line rTv be drawn parallel to ECO, Rr will $=Vv$, and $Er=Ov=CT$. Hence

$$ER+OV=Er-Rr+Ov+Vv=2CT;$$

doubling both sides,

$$EG+OP, \text{ or } EG+EH=2CK=CK+CL.$$

4. Prop. 3. If the angles made by EG and EH with EF be increased till the point G falls on the other side of E, as in fig. 3, then

$$EH-EG=CK+CL.$$

Making the same construction, we find

$$OV-ER=Ov+Vv-(Rr-Er)=2CT,$$

whence $OP-EG$, or $EH-EG$, $=2CK=CK+CL$.

5. Prop. 4. The vertical solid angle of a homogeneous pyramid being given, its attraction upon a particle placed at the vertex, is proportional to its length, $\left(\text{the force to each particle} \propto \frac{\text{mass}}{(\text{distance})^2}\right)$. For conceive the pyramid to be divided into an indefinitely great number of strata of the same thickness, by sections perpendicular to its axis: the homologous sides of these sections will be as the distance from the vertex; therefore the areas of the sections will be as the square of that distance; and therefore the mass included between two sections will be ultimately as the square of that distance. But the attraction on a particle at the vertex, is as the mass directly, and the square of the distance inversely. Hence, the attraction of every stratum is the same; and, consequently, the whole attraction of the pyramid will be proportional to the number of strata, that is, to its length.

In the same manner it appears, that the attraction of a frustrum of the same pyramid upon a particle placed at the vertex of the pyramid, is as the length of the frustrum.

6. Prop. 5. If the base of a pyramid, whose vertical solid angle is small, be given, the attraction on a point in the vertex $\propto \frac{\text{base}}{\text{length}}$.

For if b be the base, l the length, x the distance of any section from the vertex, the area of this section $= \frac{bx^2}{l^2}$; hence, the mass included between the section at the distance x, and that at the distance $x+\delta x$ ultimately $= \frac{bx^2\delta x}{l^2}$; and its attraction $= \frac{b\delta x}{l^2}$; putting u for the attraction, $\frac{du}{dx}=\frac{b}{l^2}$; $\therefore u=\frac{bx}{l^2}$, which for the whole pyramid $= \frac{b}{l}$.

7. If we put k for the density of the matter, that is, if the attraction of the matter in the volume M, on a point at the distance D, produce an accelerating force $= \frac{Mk}{D^2}$, then the attraction of one stratum ultimately $= \frac{kb\delta x}{l^2}$, whence $u = \frac{kbx}{l^2}$, or for the whole pyramid $= \frac{kb}{l}$.

ON THE ATTRACTION OF AN OBLATE SPHEROID.

8. Prop. 6. To find the attraction of an oblate spheroid on a particle placed at its pole.

Let B, (fig. 4.), be the pole of the spheroid, BD the axis; let the spheroid be divided into wedges, by planes passing through BD, two of which are BPD, BQD, making with each other the very small angle ω; in these planes draw BP, BQ making with BD the angle θ, and Bp, Bq, making with BP, BQ the very small angle $\delta\theta$; and suppose the wedge divided into pyramids similar to BPq. Let x be the abscissa of P, measured along the axis of the spheroid; y its ordinate; let $BP = r$. If through qp a section $pqts$ be drawn perpendicular to the axis of the pyramid, since qp ultimately $= y\omega$, and $qt = r\theta$; the area of this section $= ry\omega\delta\theta$; therefore by (7), the attraction of the pyramid $= \frac{kry\omega\delta\theta}{r} = ky\omega\delta\theta$. This is in the direction BP: but as the whole attraction of the spheroid will evidently be in the direction BD, we must resolve the attraction of the pyramid into one parallel to BD, and one perpendicular to BD: the former will be effective, but the latter will be counteracted by forces in the opposite direction. The effective part

$$= ky\omega\delta\theta \times \frac{x}{r} = kr \,.\, \sin\theta \,.\, \cos\theta \,.\, \omega \,.\, \delta\theta.$$

Let a be the equatoreal radius of the spheroid, b the semi-axis;

$$\text{then} \quad y^2 = \frac{a^2}{b^2}(2bx - x^2).$$

Putting for x and y their values $r\cos\theta$ and $r\sin\theta$, this becomes

$$r^2\sin^2\theta = \frac{a^2}{b^2}(2br\cos\theta - r^2\cos^2\theta),$$

$$\text{whence } r = \frac{2b\cos\theta}{1 - \frac{a^2 - b^2}{a^2}\sin^2\theta} = \frac{2b\cos\theta}{1 - e^2\sin^2\theta}$$

putting e for the eccentricity of the generating ellipse. Hence, the attraction of the pyramid ultimately

$$= k\omega \times \frac{2b\cos^2\theta \,.\, \sin\theta \,.\, \delta\theta}{1 - e^2\sin^2\theta}$$

and if w be the attraction of the wedge,

$$\frac{dw}{d\theta} = 2kb\omega \times \frac{\cos^2\theta \,.\, \sin\theta}{1 - e^2\sin^2\theta}.$$

Let $\cos\theta = z$; then

$$\frac{dw}{dz} = -2kb\omega \times \frac{z^2}{1 - e^2 + e^2z^2};$$

integrating,

$$w = -\frac{2kb\omega}{e^2} \times \left(z - \frac{\sqrt{1-e^2}}{e} \,.\, \tan^{-1}\frac{ez}{\sqrt{1-e^2}}\right).$$

Taking this from $\theta=0$, to $\theta=\frac{\pi}{2}$, or from $z=1$, to $z=0$,

$$w=2kb\omega\left(\frac{1}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\tan^{-1}\frac{e}{\sqrt{1-e^2}}\right).$$

This is the attraction of a wedge whose angle is ω; and since the attraction of every wedge with an equal angle must be the same, the attraction of the whole spheroid will be found by putting 2π in the place of ω; that is, the attraction

$$=4\pi.kb\left(\frac{1}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\tan^{-1}\frac{e}{\sqrt{1-e^2}}\right)$$

$$=4\pi.kb\left(\frac{1}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e\right).$$

9. If the spheroid differs very little from a sphere, we may put $a=b\,(1+\epsilon)$; ϵ is then called the ellipticity of the spheroid. Then $e^2=2\epsilon$, nearly. Hence, the attraction of the pyramid

$$=2bk\omega\frac{\cos^2\theta\sin\theta\delta\theta}{1-2\epsilon\sin^2\theta}$$

$$=2bk\omega\,(\cos^2\theta.\sin\theta.\delta\theta+2\epsilon.\cos^2\theta\sin^3\theta.\delta\theta),\text{ nearly}$$

$$=2bk\omega\,(\cos^2\theta+2\epsilon\cos^2\theta-2\epsilon\cos^4\theta)\sin\theta.\delta\theta;$$

whence, the attraction of the wedge

$$=2bk\omega\left(-\frac{\cos^3\theta}{3}-\frac{2\epsilon}{3}\cos^3\theta+\frac{2\epsilon}{5}\cos^5\theta\right);$$

and taking this integral from $\theta=0$, to $\theta=\frac{\pi}{2}$, the attraction of the wedge

$$=2bk\omega\left(\frac{1}{3}+\frac{2\epsilon}{3}-\frac{2\epsilon}{5}\right)=\frac{2}{3}bk\omega\left(1+\frac{4\epsilon}{5}\right).$$

Then the attraction of the whole spheroid is found as before, by putting 2π for ω, and is, therefore,

$$= \frac{4\pi}{3} kb \left(1 + \frac{4e}{5}\right).$$

10. Prop. 7. To find the attraction of an oblate spheroid on a particle at its equator.

Let ARM, fig. 5, be the equator of the spheroid; AZ a perpendicular to it from the attracted point A; suppose the spheroid divided into wedges, by planes passing through AZ; let two of these planes, very near each other, be APR, Apr; and drawing the diameter AM, let $MAR = \phi$, $MAr = \phi + \delta\phi$. Then suppose the lines AP, Ap, to be drawn in the planes APR, Apr, making with AR, Ar, equal angles θ, and AQ, Aq to be very near them, making with AR, Ar, equal angles $\theta + \delta\theta$. If through q we draw a plane $qtsw$ perpendicular to the axis of the small pyramid APq,

$$wq \text{ ultimately } = r \cos\theta\delta\phi, \quad qt = r\delta\theta,$$

and the base of the pyramid

$$= wq \times qt = r^2 \cos\theta\delta\phi\delta\theta;$$

therefore by (7), its attraction in the direction AO

$$= \frac{kr^2 \cos\theta\delta\phi . \delta\theta}{r} = kr \cos\theta\,\delta\phi . \delta\theta.$$

Draw PN perpendicular to the plane of the equator: NO perpendicular to AM; let $AO = x$, $ON = y$, $NP = z$. If we resolve the attraction of the pyramid into two parts, one in the direction AN, and the other in NP, the latter of these will be counteracted by the attraction of another pyramid in the same wedge, and the former

$$= k . r . \cos\theta . \delta\phi . \delta\theta . \cos\theta = kr \cos^2\theta . \delta\phi\delta\theta.$$

This is in the direction AN; if we resolve it into two in the

directions AO, ON, the latter of these will be counteracted by the attraction of an equal pyramid in another wedge, making the same angle with AM; and the former

$$=k\,.\,r\,.\cos^2\theta\,.\,d\phi\,.\,\delta\theta \times \cos\phi = k\,.\,r\,.\cos\phi\,.\cos^2\theta\,.\,\delta\phi\,.\,\delta\theta.$$

Now the equation to the spheroid is

$$PN^2 = \frac{b^2}{a^2}(AC^2 - CO^2 - ON^2),$$

$$\text{or } z^2 = \frac{b^2}{a^2}(2ax - x^2 - y^2);$$

putting for x, y, and z, their values

$$r\,.\cos\theta\,.\cos\phi,\quad r\,.\cos\theta\,.\sin\phi,\quad r\sin\theta,$$

it becomes

$$r^2\sin^2\theta = \frac{b^2}{a^2}(2ar.\cos\theta.\cos\phi - r^2\cos^2\theta\cos^2\phi - r^2\cos^2\theta\sin^2\phi);$$

$$\therefore r = \frac{2b^2}{a}\,.\,\frac{\cos\theta\,.\cos\phi}{1 - e^2.\cos^2\theta + \sin^2\theta}.$$

Hence, the effective attraction of the pyramid

$$= \frac{2kb^2}{a}\,.\,\frac{\cos^2\phi\,.\cos^3\theta\,.\,\delta\phi\,.\,\delta\theta}{1 - e^2.\cos^2\theta + \sin^2\theta}.$$

Let w be the effective attraction of the wedge;

$$\frac{dw}{d\theta} = \frac{2kb^2}{a}\cos^2\phi\,.\,\delta\phi\,.\,\frac{\cos^3\theta}{1 - e^2.\cos^2\theta + \sin^2\theta}.$$

To integrate this, let $\sin\theta = v$;

$$\frac{dw}{dv} = \frac{2kb^2}{a}\,.\cos^2\phi\,.\,\delta\phi\,.\,\frac{1 - v^2}{1 - e^2 + e^2v^2};$$

$$\therefore\ w = \frac{2kb^2}{a}\cos^2\phi\,.\,\delta\phi$$

$$\left(-\frac{v}{e^2} + \frac{1}{e^3\sqrt{1-e^2}}\,.\,\tan^{-1}\frac{ev}{\sqrt{1-e^2}}\right).$$

Taking this from $\theta = -\frac{\pi}{2}$, to $\theta = +\frac{\pi}{2}$, or from $v = -1$, to $v = +1$,

$$w = \frac{2kb^2}{a}\cos^2\phi\,.\,\delta\phi\,.\left(\frac{2}{e^3\sqrt{1-e^2}}\,.\,\tan^{-1}\frac{e^2}{\sqrt{1-e^2}} - \frac{2}{e^2}\right)$$

$$= 4kb\cos^2\phi\,.\,\delta\phi \times \left(\frac{1}{e^3}\sin^{-1}e - \frac{\sqrt{1-e^2}}{e^2}\right).$$

Hence, if u be the attraction of the spheroid,

$$\frac{du}{d\phi} = 4kb\cos^2\phi\left(\frac{1}{e^3}\sin^{-1}e - \frac{\sqrt{1-e^2}}{e^2}\right);$$

integrating from $\phi = -\frac{\pi}{2}$, to $\phi = +\frac{\pi}{2}$,

$$u = 2kb\pi\left(\frac{1}{e^3}\sin^{-1}e - \frac{\sqrt{1-e^2}}{e^2}\right).$$

11. If the spheroid differ little from a sphere, putting, as before, $a = b\,(1+\epsilon)$, we find

$$e^2 = 2\epsilon,\quad \frac{b^2}{a} = b\,(1-\epsilon),$$

$$\text{and}\quad \frac{dw}{dv} = 2kb\,(1-\epsilon)\cos^2\phi\,.\,\delta\phi\,.\,\frac{1-v^2}{1-2\epsilon+2\epsilon v^2}$$

$$= 2kb\,.\,(1-\epsilon)\,.\cos^2\phi\,.\,\delta\phi\,.\,(1-v^2+2\epsilon-4\epsilon v^2+2\epsilon v^4),\ \text{nearly},$$

$$\text{and}\quad w = 2kb\,(1-\epsilon)\,.\cos^2\phi.\delta\phi.\left(v - \frac{v^3}{3} + 2\epsilon v - \frac{4\epsilon v^3}{3} + \frac{2\epsilon v^5}{5}\right),$$

which, from $v = -1$, to $v = +1$

$$= 4kb\,(1-\epsilon)\cos^2\phi\,.\,\delta\phi\,.\left(\frac{2}{3}+\frac{16\epsilon}{15}\right)$$

$$8kb\,(1-\epsilon)\cos^2\phi\,.\,\delta\phi\,.\left(1+\frac{8}{5}\epsilon\right)$$

$$8kb\,\cdot\cos^2\phi\,.\,\delta\phi\left(1+\frac{3}{5}\epsilon\right);$$

$$\therefore\ \frac{du}{d\phi}=\frac{8kb}{3}\cos^2\phi\left(\ \ +\frac{3}{5}\epsilon\right);$$

therefore integrating from $\phi = -\frac{\pi}{2}$ to $\phi = +\frac{\pi}{2}$,

$$u=\frac{4kb\pi}{3}\left(1+\frac{3}{5}\epsilon\right).$$

12. PROP. 8. If E, (fig. 6.), be any point on the surface of a spheroid, and EC be drawn perpendicular to the plane of the equator, and a spheroid be described concentric, similar, and similarly situated to the given spheroid, touching EC at C; then the attraction of the given spheroid on E, in a direction parallel to the radius CW, is equal to the attraction of the smaller spheroid in the same direction on the point C.

13. Suppose both spheroids divided into wedges by planes passing through EC: let $EGHB$, $CKDL$, be the sections of both, made by one plane; and $ERSb$, $CTdV$, the sections made by another plane very near the former. Draw EF parallel to CD, and let the angles GEF, HEF, KCD, LCD, be all equal. Let the angles gEF, hEF, kCD, lCD be also equal to each other, and very nearly equal to the former; and suppose the wedge divided into pyramids by planes passing through these lines perpendicular to the plane AEB. Then the angles GEg, HEh, KCk, LCl, are equal. And therefore since the axes of the pyramids GEr, HEs, KCt, LCv,

are equally inclined to EC, the edge of the wedge, their solid angles will be equal. Consequently, by Prop. 4., their attractions in the directions of their axes will be as their lengths. And the attraction of each in the direction EF or CD will be as its length multiplied by the cosine of the angle, which its axis makes with EF or CD; or since this angle is the same for all, the attraction of each in the direction EF or CD will be as its length. Hence the sum of the attractions of EGr and EHs in direction EF : sum of attractions of CKt and CLv in direction CD :: $EG + EH$: $CK + CL$. But, by Prop. 1., AEB and CKD are similar and concentric ellipses; therefore by Prop. 2.

$$EG + EH = CK + CL;$$

or the attractions of GEr and HEs in direction $EF =$ attractions of KCt, LCv, in direction CD. If the point G had fallen on the other side of E, we should have had, by Prop. 3.

$$EH - EG = CK + CL,$$

and therefore the difference of the attractions of the two pyramids, whose vertices are at E, $=$ to the sum of the attractions of those whose vertices are at C; but since in this case the resolved part of the attraction of EGr is in a direction opposite to EF, we may still say, that the attractions of GEr, HEs, in direction EF $=$ attractions of KCt, LCv, in direction CD. And the same is true for all other corresponding pairs of pyramids. Now since the angle $GEg = KCk$, and $HEh = LCl$, by taking the same number of pairs of pyramids, we shall at the same time have taken the whole double wedge $AEBO$, and the whole wedge $CKDL$; and for every corresponding pair of pyramids, the attraction in direction CD or EF is the same; therefore the attraction of E by the double wedge $AEBO$ in direction EF, is the same as the attraction of C by the wedge $CKDL$, in direction CD.

Now resolve each of these attractions into two, one perpendicular to the axis of the spheroid, another at right angles to this. Since EC is parallel to the axis of the spheroid, the perpen-

diculars upon the axis from E and C will be parallel; but EF and CD are parallel; therefore the angle made by EF with the perpendicular from E, is equal to the angle made by CD with the perpendicular from C. Consequently, the resolved parts of the equal attractions in directions perpendicular to the axis of the spheroid, will also be equal. Now the double wedge $AEBO$, and the wedge $CKDL$ are formed by the same planes; and therefore the number of wedges into which the two spheroids can be cut is the same; and since the attractions of each corresponding pair of wedges, in direction perpendicular to the axis, is the same, the attractions of the whole spheroids in that direction will be the same; or the attraction of the larger spheroid on E, in direction perpendicular to the axis, is equal to the attraction of the smaller spheroid on C, in the same direction.

14. PROP. 9. To find the attraction on E, in a direction perpendicular to the axis of the spheroid.

By the last proposition, this is equal to the attraction of the spheroid CN in the same direction on the point C. And by Prop. 7, the attraction of a spheroid, whose polar and equatoreal radii are b and a, on a point in its equator

$$=2kb\pi\left(\frac{1}{e^3}\sin^{-1}e-\frac{\sqrt{1-e^2}}{e^2}\right)$$

$$=2ka\pi\left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e-\frac{1-e^2}{e^2}\right).$$

Consequently, the attraction on E, in the direction perpendicular to the axis of the spheroid

$$=CW\times 2k\pi\times\left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e-\frac{1-e^2}{e^2}\right),$$

where e is the eccentricity of the smaller spheroid, which is the same as that of the larger, as the spheroids are similar. In the same spheroid, it will be observed, this is proportional to CW.

15. PROP. 10. If from any point E, (fig. 7.) on the surface of a spheroid, a line EX be drawn perpendicular to the axis,

and a spheroid XY be described concentric, similar, and similarly situated, to the given spheroid, touching that line at its pole X; then the attraction of the given spheroid on the point E, in a direction parallel to its axis, is equal to the attraction of the smaller spheroid on a point at its pole X.

The demonstration of this is, in all respects, similar to the demonstration of Prop. 8. The spheroids must be divided into wedges, by planes passing through the line EX, and the sections of the spheroids made by one plane, will be similar and concentric ellipses.

16. Prop. 11. To find the attraction on E, in a direction parallel to the axis of the spheroid.

By Prop. 10, this is equal to the attraction of the spheroid XY, on a particle at X. But, by Prop. 6., the attraction of the spheroid XY, on a particle at X

$$= 4\pi k \, . \, WX \, . \left(\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3} \sin^{-1} e\right);$$

therefore the attraction of the larger spheroid on E. in a direction parallel to its axis, is

$$4\pi k \, . \, EC \, . \left(\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3} \sin^{-1} e\right).$$

In the same spheroid this is proportional to EC.

17. Prop. 12. If a point E, (fig. 8.) be placed at the interior surface of a shell, bounded by similar and concentric spheroidical surfaces, the attraction of the whole shell will be 0.

For suppose the small pyramids EF, EG, to be formed by the same planes passing through E; let a plane pass through the axis of the pyramids, and through the common center of the spheroids; through H the point of bisection of EM, draw $WHKL$. Since EM is bisected in H, the tangent at K is parallel to EM; and since the ellipses are similar and concentric, the tangent at L is parallel to that at K; it is therefore parallel

to FG; and FG therefore is bisected in H, or $FH = HG$. But $EH = HM$; $\therefore$ $EF = MG$. Now the attractions of the pyramid FE, and the frustrum MG, upon a point at E, are proportional to their lengths EF and MG, by Prop. 4.; they are, therefore, equal; and they are in opposite directions; therefore they destroy each other. Now the whole shell may be divided into pairs of pyramids, in each of which it may be shewn, that the attraction is 0; therefore the attraction of the whole shell $= 0$.

18. Prop. 13. To find the attraction of a spheroid on a point within it.

From E the point, draw EC perpendicular to the plane of the equator. By Prop. 12, the attraction of the shell external to the spheroid EKM is 0; and by Prop. 9., the attraction of the spheroid EKM on E, in the direction perpendicular to the axis of the spheroid, is

$$CW \times 2k\pi . \left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1} e - \frac{1-e^2}{e^2}\right).$$

By Prop. 11., the attraction in direction parallel to the axis, is

$$EC \times 4k\pi \left(\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3}\sin^{-1} e\right).$$

19. The results of all these propositions may be thus stated. The attraction of any particle of a spheroid perpendicular to the axis equals its distance from the axis $\times$ Q, where

$$Q = 2k\pi \left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1} e - \frac{1-e^2}{e^2}\right).$$

The attraction perpendicular to the plane of the equator equals its distance from the plane of the equator $\times$ P, where

$$P = 4k\pi \left(\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3}\sin^{-1} e\right).$$

If the spheroid differ little from a sphere,

$$Q = \frac{4k\pi}{3}\left(1 - \frac{2}{5}\epsilon\right), \quad P = \frac{4k\pi}{3}\left(1 + \frac{4}{5}\epsilon\right).$$

These expressions may be found by expanding the values just given: or by making the attractions at the pole and equator coincide with those found in (11) and (9).

APPLICATION OF THESE THEOREMS TO THE FIGURE OF THE EARTH.

20. In investigating the figure of the Earth, we shall suppose that the Earth was originally a homogeneous fluid mass, every particle attracting every other particle with an accelerating force, proportional to the mass of the attracting particle directly, and the square of the distance of the attracted particle inversely. This mass we suppose to revolve about an axis in $23^h.\ 56'.\ 4''$.

21. Now if the Earth had no motion of rotation, it would evidently assume a spherical figure. For the mutual attraction of the particles would collect the whole into one mass; and if any one part were then protuberant above the rest, the direction of gravity would not be perpendicular to its surface, and it would not remain in that form, but would run down, (Vince's *Hydrostatics*, Prop. 3.) The form then must be such as would leave no part protuberant above the rest; that is, it must be spherical.

22. But, in consequence of the Earth's rotation, every particle has a centrifugal force, or a tendency to recede from the axis of rotation. The effect of this, it is plain, will be to enlarge the Earth at its equator, and to flatten it at the poles. It is our object now to shew, that the figure which the Earth would assume, is accurately that of an oblate spheroid.

23. To prove this we shall shew that, upon giving a proper value to the ellipticity, the whole force which acts upon any point at the surface is perpendicular to the surface; and if two

canals of any form be made in the fluid, terminated at any points in the surface, and leading to the same point in the interior, the pressure on this point is the same from the fluid in both canals.

24. Let T be the time of revolution: the centrifugal force on the particle E, fig. 6, 7, and 8, is

$$\frac{4\pi^2}{T^2} EX, \quad \text{or} \quad \frac{4\pi^2}{T^2} CW,$$

and is in the direction XE. Adding this to the forces mentioned in (19), we have, the whole force acting upon the point E in direction $EX = \left(Q - \frac{4\pi^2}{T^2}\right) CW$: that in direction $EC = P \,.\, EC$. The forces therefore upon any point, estimated in directions perpendicular to the axis, and perpendicular to the equator, are still proportional to the distances from the axis, and from the plane of the equator.

25. PROP. 14. That the fluid in a canal from the equator to the center, and that in a canal from the pole to the center, may produce the same pressure on a particle at the center, the whole force at the pole must be to that at the equator as the radius of the equator to the radius of the pole.

Let p be the distance of any point in the polar canal from the center; $p + \delta p$ the distance of a point near it. The pressure which is produced by the fluid included between these, is proportional to the quantity of the fluid multiplied by the accelerating force that acts on it, and is therefore ultimately proportional to $\delta p \times Pp$; or (supposing the section of the canal $= 1$) it ultimately $= Pp \,.\, \delta p$. Let u be the whole pressure; since, upon increasing the distance by δp, the pressure is diminished by $Pp \,.\, \delta p$, we have

$$\frac{du}{dp} = -Pp; \quad \therefore u = C - \frac{Pp^2}{2}.$$

But the pressure at the surface $= 0$,

$$\text{or } C - \frac{Pb^2}{2} = 0; \quad \therefore u = \frac{P}{2}(b^2 - p^2):$$

hence, the pressure at the center $= \frac{Pb^2}{2}$. Similarly the pressure at the center, produced by the equatoreal column

$$= \left(Q - \frac{4\pi^2}{T^2}\right)\frac{a^2}{2}.$$

When the pressures are equal (which, from the nature of fluids is necessary for equilibrium),

$$Pb^2 = \left(Q - \frac{4\pi^2}{T^2}\right)a^2,$$

$$\text{or } P.b : \left(Q - \frac{4\pi^2}{T^2}\right)a :: a : b.$$

But $Pb =$ force at the pole : $\left(Q - \frac{4\pi^2}{T^2}\right)a =$ the whole force at the equator; therefore the force at the pole : force at the equator :: equatoreal radius : polar radius.

26. Prop. 15. When this proportion holds, the whole force at any point on the surface is perpendicular to the surface.

Let E, fig. 9, be the point on the surface: take EC to represent the force in the direction EC, and CN to represent that in direction EX: then, EN will represent the magnitude and direction of the whole force at E. Now,

$$EC : CN :: P.EC : \left(Q - \frac{4\pi^2}{T^2}\right).CW, \text{ by (24),}$$

$$\text{or } :: a^2.EC : b^2.CW,$$

by the demonstration of Prop. 14;

$$\text{or } EC : CN :: EC : \frac{b^2}{a^2}CW.$$

Hence, $CN = \frac{b^2}{a^2} CW =$ subnormal (by *Conic Sections*); therefore, EN is the normal; that is, the whole force is perpendicular to the surface. It appears also that the whole force is represented in magnitude by the normal.

27. Prop. 16. When the same proportion holds, if to any point within the spheroid canals of any form be drawn, terminated any where in the surface; the pressure on that point, found by adding the pressures of successive portions of any of the canals, will be the same for every canal.

Let E, fig. 11, be the point, $EO F$ a canal; take O and o, two points very near each other; draw ON, on, perpendicular to the plane of the equator, and OM, om, perpendicular to the axis; draw Ol perpendicular to no, and OK perpendicular to mo: let $MO = x$, $NO = y$, $EO = s$; $mo = x + \delta x$, $no = y + \delta y$, $Eo = s + \delta s$. If we suppose the section of the canal $= 1$, the quantity of matter in the length $Oo = \delta s$. And the accelerating force in the direction $ON = P \times ON = P . y$: the resolved part of this in the direction of the canal

$$= P . y . \cos Oon = P . y . \frac{\delta y}{\delta s} = \text{ultimately } P . y . \frac{dy}{ds}.$$

And the accelerating force in the direction OM

$$= \left(Q - \frac{4\pi^2}{T^2}\right) . OM = \left(Q - \frac{4\pi^2}{T^2}\right) x:$$

the resolved part, in the direction of the canal, is found by multiplying it by $\cos Ook$, that is, by $\frac{ok}{Oo}$ or $\frac{\delta x}{\delta s}$, or ultimately by $\frac{dx}{ds}$: hence, the whole accelerating force in that direction

$$= Py \frac{dy}{ds} + \left(Q - \frac{4\pi^2}{T^2}\right) . x . \frac{dx}{ds}:$$

and the pressure produced by the action of this force on the fluid in Oo

$$=\delta s \times \left(P y \frac{dy}{ds} + \overline{Q - \frac{4\pi^2}{T'^2}} \cdot x \frac{dx}{ds}\right).$$

This, if we put p for the pressure, is the decrement of p, produced by increasing s by δs: hence, $\dfrac{dp}{ds} = \dfrac{\delta p}{\delta s}$ ultimately

$$-P y \frac{dy}{ds} - Q - \frac{4\pi^2}{T'^2} \cdot x \frac{dx}{ds}$$

$$\text{Integrating, } p = C - \frac{P y^2}{2} - \overline{Q - \frac{4\pi^2}{T'^2}} \cdot \frac{x^2}{2}.$$

Let the values of x and y, at the point E, be f and g; and where the canal meets the surface, let the values of x and y be v and w: then, observing that the pressure at the surface is $=0$, we find the pressure at E

$$= \frac{P w^2}{2} + \overline{Q - \frac{4\pi^2}{T'^2}} \cdot \frac{v^2}{2} - \frac{P g^2}{2} - \overline{Q - \frac{4\pi^2}{T'^2}} \cdot \frac{f^2}{2}.$$

But, by the equation to the generating ellipse,

$$w^2 = \frac{b^2}{a^2}(a^2 - v^2); \quad \therefore \ \frac{P w^2}{2} = \frac{P b^2}{2} - \frac{P b^2}{a^2} \cdot \frac{v^2}{2}:$$

and, by the demonstration of Prop. 14,

$$\frac{P b^2}{a^2} = Q - \frac{4\pi^2}{T'^2}.$$

Hence, the pressure at E

$$= \frac{P b^2}{2} - \frac{P g^2}{2} - Q - \frac{4\pi^2}{T^2} \cdot \frac{f^2}{2}.$$

This expression, it may be remarked, is independent of the form of the canal, and of the place at which it terminates in the

surface, and therefore we should have found the same for the pressure produced by the fluid in any other canal, as GE. For all canals therefore leading to the same point, the pressure on that point is the same.

28. From Prop. 15, we find, that if a fluid mass have the form of an oblate spheroid, there will be no tendency to disturb the particles at the surface; and from Prop. 16. it appears that, as the pressure on every particle is equal in all directions, none of the interior particles will have any tendency to motion. Every part therefore will be at rest: and therefore the oblate spheroid is the form of equilibrium, if the force at the pole : whole force at the equator :: equatoreal axis : polar axis.

29. PROP. 17. To find the proportion of the axes of the spheroid, which is in equilibrium.

The force at the pole, by Prop. 6,

$$=4\pi . kb\left(\frac{1}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e\right).$$

The attraction at the equator, by Prop. 7,

$$=2\pi . kb\left(\frac{1}{e^3}\sin^{-1}e-\frac{\sqrt{1-e^2}}{e^2}\right);$$

the centrifugal force there

$$=\frac{4\pi^2}{T^2}a=\frac{4\pi^2}{T^2}.\frac{b}{\sqrt{1-e^2}}:$$

hence, the whole force at the equator

$$=2\pi . kb\left(\frac{1}{e^3}\sin^{-1}e-\frac{\sqrt{1-e^2}}{e^2}\right)-\frac{4\pi^2}{T^2}.\frac{b}{\sqrt{1-e^2}}.$$

These must be in the proportion of $a : b$, or 1 :
that is,

$$2k\left(\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e\right) : k\left(\frac{1}{e^3}\sin^{-1}e - \frac{\sqrt{1-e^2}}{e^2}\right)$$

$$- \frac{2\pi}{T^2} \cdot \frac{1}{\sqrt{1-e^2}} :: 1 : \sqrt{1-e^2}.$$

Let $q = \dfrac{2\pi}{kT^2}$: substituting and reducing,

$$\frac{3(1-e^2)}{e^2} - \frac{(3-2e^2)\sqrt{1-e^2}}{e^3}\sin^{-1}e + q = 0.$$

The solution of this equation will give e, when q is known.

30. As this is a transcendental equation, it can be solved only by approximation; but some properties of its roots may be found thus. The left side of the equation is positive, when $e = 0$, and when $e = 1$, which are the extreme values of e that can be admitted; and it has therefore no roots, or an even number. And by constructing a curve, as fig. 10, in which the abscissa is e, and the ordinate is proportionate to the value of the first side of the equation, we find that the curve cannot cut the axis in more than two points, and therefore there can be but two forms of the oblate spheroid, which are figures of equilibrium. If, in one of these forms e be small, in the other it will be very nearly $= 1$.

31. To find these forms on the supposition that the centrifugal force is small, or q small, we will begin with supposing e small. In this case it will be most convenient to use the formulæ of (9) and (11). Then,

$$\frac{4\pi}{3}bk\left(1 + \frac{4\epsilon}{5}\right) : \frac{4\pi}{3}bk\left(1 + \frac{3\epsilon}{5}\right)$$

$$-2\pi kqb(1+\epsilon) :: 1+\epsilon : 1;$$

or, neglecting $q\epsilon$, which is the product of two small quantities,

$$1 + \frac{4\epsilon}{5} : 1 + \frac{3\epsilon}{5} - \frac{3q}{2} :: 1 + \epsilon : 1.$$

$$\text{Hence, } 1 + \frac{4\epsilon}{5} = 1 + \frac{8\epsilon}{5} - \frac{3q}{2}, \text{ or } \epsilon = \frac{5}{4} \cdot \frac{3}{2} q.$$

32. It is convenient to express the ellipticity in terms of the proportion of the centrifugal force at the equator to gravity. This proportion is

$$\frac{2\pi k b q}{\frac{4\pi}{3} bk\left(1 + \frac{3\epsilon}{5}\right) - 2\pi k q} \qquad \frac{2\pi k b q}{\frac{4\pi}{3} bk} \text{ nearly} = \frac{3}{2} q:$$

let this $= m$. Then $\epsilon = \frac{5m}{4}$.

33. To find the other form of equilibrium, we observe that e is nearly $= 1$, and therefore $\frac{b}{a}$ is small. We must therefore expand the terms of the equation in powers of $\frac{b}{a}$. For e^2 put $1 - \frac{b^2}{a^2}$, or $1 - c^2$, where $c = \frac{b}{a}$, and it becomes

$$\frac{3c^2}{1-c^2} - \frac{(1+2c^2)c}{(1-c^2)^{\frac{3}{2}}} \cos^{-1} c + q = 0.$$

Taking only the first power of c, we have

$$-c \cdot \frac{\pi}{2} + q = 0, \text{ or } \frac{b}{a} = c = \frac{2q}{\pi}.$$

34. In the earth it is found that $m = \frac{1}{289}$, or $q = \frac{1}{434}$: hence, supposing the ellipticity small, $\epsilon = \frac{1}{230}$: or, supposing the eccentricity nearly $= 1$, $c = \frac{1}{681}$. That is, the earth would be an oblate spheroid, with axes either in the proportion of 230 : 231, or in the proportion of 1 : 681. It is found by

measurement, that the ratio of the axes is nearly 300 : 301 : hence, the earth is not homogeneous.

35. In the spheroid of small ellipticity, the proportion of gravity at the pole to that at the equator, is the same as the ratio of the axes, or is the ratio 231 : 230, supposing the earth homogeneous. By observation, it is found to be about 188 : 187.

36. Resuming the consideration of the general equation of (29), and the construction of (30), it is easily seen that upon giving to q a certain value, the curve will touch the line of abscissæ: and upon increasing q the curve will not meet the line of abscissæ at all. In the former case, then, there is but one form of equilibrium, and, in the latter, equilibrium is not possible. To find e', the value of e, which gives but one form, we may observe, that two roots of the equation have become coincident, or are equal: if then we take the differential coefficient of the first side of the equation, it must have one of the equal roots: or the same value of e will make it $= 0$. This gives

$$-\frac{9}{e'^3} + \frac{2}{e'} + \left(\frac{9}{e'^4} - \frac{8}{e'^2}\right) \cdot \frac{\sin^{-1} e'}{\sqrt{1 - e'^2}} = 0:$$

solving this equation by approximation, $e' = ,92995$, whence $\frac{b'}{a'} = ,36769$, $\frac{a'}{b'} = 2.7197$. Substituting this value of e in the general equation, $q' = ,224671$. And, if T' be the time of revolution, since $q' = \frac{2\pi}{kT'^2}$, we have

$$T' = \sqrt{\frac{2\pi}{kq'}} = \frac{5,2883}{\sqrt{k}}.$$

37. In a fluid, whose density is the same as the density of the earth, supposed homogeneous,

$$\frac{1}{434} = \frac{2\pi}{k\,(23^{\text{h}}.\,56'.\,4'')^2}, \text{ by (34).}$$

Dividing this by the equation $q' = \frac{2\pi}{kT'^2}$, we have

$$\frac{1}{2 \cdot 434 \cdot q'} = \left(\frac{T'}{23^h.\,56'.\,4''}\right)^2,$$

$$\text{or } T' = 23^h.\,56'.\,4'' \times \sqrt{\frac{1}{434 \cdot q'}} = 2^h.\,25'.\,26''.$$

A spheroid then cannot remain in equilibrium, if it revolve in a shorter time than $2^h.\,25'.\,26''$, its density being the same as that of the earth.

38. The expression for the ellipticity in (31), gives us the means of comparing the ellipticities of different planets, supposed homogeneous. For the ellipticity

$$= \frac{15}{8}\, q = \frac{15}{4} \cdot \frac{\pi}{kT'^2}.$$

If then we can in any manner compare the masses of the planets, (which can be done immediately with those that have satellites,) and, if we know the ratio of their diameters, the ratio of their densities will be known: and, knowing also the ratio of their times of revolution, their ellipticities will be inversely as their densities × the square of the times of revolution.

ON THE FIGURE OF THE EARTH, SUPPOSING IT HETEROGENEOUS.

39. The results which we have deduced relative to the figure of the Earth, supposing it homogeneous, do not agree with observation: the homogeneity of the Earth is also, *a priori*, very improbable. We shall now proceed to shew that, sup-

posing the Earth heterogeneous, a spheroidal form, of ellipticity different from that which we have found, will be a form of equilibrium. As before, we shall suppose that the Earth was originally a fluid mass: and we shall consider the density of different parts to be different, either from its original constitution, or from the difference of pressure produced by the weight of the superincumbent mass. From the difficulty of the investigation, we are obliged to suppose the ellipticity small, and to reject all quantities depending on its square and higher powers.

40. Prop. 18. If the base of a prism be very small, to find its attraction in the direction of its axis on a point any where without it.

Let BC, (fig. 12.) be the given prism: A the given point: draw AD perpendicular to the axis, produced if necessary; take two sections perpendicular to the axis, passing through the points E and F, which are very near each other: join AB, AE, AC. Let $AD=a$, $DB=b$, $DC=c$, the section of the prism $=S$, its density $=\rho$: $DE=x$, $DF=x+\delta x$. The mass included between the two sections through E and $F=S\,.\,\delta x$; therefore its attrrction on $A=\dfrac{\rho\,.\,S\,.\,\delta x}{(AE)^2}$ ultimately; therefore the resolved part in the direction of the prism's axis

$$=\frac{\rho\,.\,S\,.\,\delta x}{(AE)^2}\times\frac{DE}{AE}=\frac{\rho\,.\,S\,.\,DE\,.\,\delta x}{(AE)^3}:$$

hence, if u be the whole attraction, $\dfrac{du}{dx}=$ ultimate value of $\dfrac{\delta u}{\delta x}$

$$=\frac{\rho\,.\,S\,.\,x}{(a^2+x^2)^{\frac{3}{2}}};\ \text{integrating, } u=-\frac{\rho\,.\,S}{(a^2+x^2)^{\frac{1}{2}}}+C.$$

Determining C so as to make $u=0$ when $x=DB=b$, we find the whole attraction

$$=\rho\,.\,S\,.\left(\frac{1}{\sqrt{a^2+b^2}}-\frac{1}{\sqrt{a^2+c^2}}\right)=\rho\,.\,S\,.\left(\frac{1}{AB}-\frac{1}{AC}\right).$$

41. PROP. 19. To find the expression upon whose integration depends the attraction of an ellipsoid on a point any where without it.

Let CD, CE, CF, be the three semi-axes of the ellipsoid, $= a, b, c$, respectively: take these directions for the directions of x, y, and z: then, the equation to the surface of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let the co-ordinates of the point A be f, g, h. Suppose the ellipsoid divided into slices by planes, as PMQ, pmq, parallel to the plane of yz: and suppose these slices divided into prisms by planes, as PNQ, pnq, parallel to the plane of xz. Let

$$CM = x,\quad Mm = \delta x,\quad MN = y,\quad mn = y + \delta y,\quad PN = z.$$

Then the parallelogram $Nn = \delta x . \delta y$. By the last Proposition therefore the attraction of the prism Pq on A, in a direction parallel to z, is

$$\rho . \delta x . \delta y \left(\frac{1}{AP} - \frac{1}{AQ}\right)$$

$$= \rho . \delta x . \delta y \left(\frac{1}{\sqrt{(\overline{f-x}|^2 + \overline{g-y}|^2 + \overline{h-z}|^2)}} - \frac{1}{\sqrt{(\overline{f-x}|^2 + \overline{g-y}|^2 + \overline{h+z}|^2)}}\right).$$

Let $x = ar$, $y = bs$; then, the attraction of the prism

$$= \rho . a . b . \delta r . \delta s \left(\frac{1}{AP} - \frac{1}{AQ}\right).$$

The attraction of the slice therefore will

$$= \rho . a . b . \delta r \times \int_s \left(\frac{1}{AP} - \frac{1}{AQ}\right),$$

taken from G to H. To find the values of y, corresponding

to those points, we must make $z=0$, in the equation to the surface: then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; \quad \therefore\ y=\pm\frac{b}{a}\sqrt{a^2-x^2}, \text{ and } s=\pm\sqrt{1-r^2}.$$

The attraction of the slice therefore

$$=\rho\,.\,a\,.\,b\,.\,\delta r\times\int_s\left(\frac{1}{AP}-\frac{1}{AQ}\right),$$

taken from $s=-\sqrt{1-r^2}$ to $s=+\sqrt{1-r^2}$.

Let this $=v\delta r$: then the attraction of the ellipsoid $=\int_r v$, taken from $x=-a$ to $x=+a$, or from $r=-1$, to $r=+1$. The attraction therefore of the ellipsoid $=\rho\,.\,a\,.\,b\times$

$$\int_r.\int_s.\left(\frac{1}{\sqrt{(\overline{f-ar}|^2+\overline{g-bs}|^2+\overline{h-ct}|^2)}}-\frac{1}{\sqrt{(\overline{f-ar}|^2+\overline{g-bs}|^2+\overline{h+ct}|^2)}}\right),$$

where $ct=z$; the first integral being taken from

$$s=-\sqrt{1-r^2} \text{ to } s=+\sqrt{1-r^2},$$

and the second from $r=-1$ to $r=+1$.

42. Prop. 20. If a spheroid, fig. 13, whose semi-axes are a, b, c, attract a point without it whose co-ordinates are la, $m\beta$, $n\gamma$, where $l^2+m^2+n^2=1$.

And if a spheroid, fig. 14, of the same density, whose semi-axes are a, β, γ, attract a point A' within it*, whose co-ordinates are la, mb, nc;

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* That A' is within the spheroid attracting it is easily shewn. For since $l^2+m^2+n^2=1$, its co-ordinates satisfy the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and

And if $a^2 - c^2 = \alpha^2 - \gamma^2$, $b^2 - c^2 = \beta^2 - \gamma^2$;

Then, the attraction on A parallel to c, is to the attraction on A' parallel to γ, as ab to $\alpha\beta$.

43. By the last Proposition, the attraction on A parallel to c

$$= \rho \cdot a \cdot b \times \int_r \int_s \left(\frac{1}{\sqrt{(\overline{l\alpha - ar}|^2 + \overline{m\beta - bs}|^2 + \overline{n\gamma - ct}|^2)}} \right.$$

$$\left. - \frac{1}{\sqrt{(\overline{l\alpha - ar}|^2 + \overline{m\beta - bs}|^2 + \overline{n\gamma + ct}|^2)}} \right).$$

The square of the denominator of the first fraction

$$= l^2\alpha^2 - 2a\alpha lr + a^2r^2$$

$$+ m^2\beta^2 - 2b\beta ms + b^2s^2$$

$$+ n^2\gamma^2 - 2c\gamma nt + c^2t^2.$$

But $l^2 + m^2 + n^2 = 1$: and the equation to the ellipsoid, or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

gives us $r^2 + s^2 + t^2 = 1$:

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and it is therefore in the surface of an ellipsoid, concentric to the ellipsoid which attracts it, and whose semi-axes are a, b, c. In the same manner, A is in the surface of an ellipsoid concentric to the given ellipsoid, whose semi-axes are α, β, γ. Now, since

$$\alpha^2 - a^2 = \beta^2 - b^2 = \gamma^2 - c^2,$$

the surfaces of the ellipsoids do not cut each other: and the point A being without the given ellipsoid, α must be $> a$, $\beta > b$, $\gamma > c$. The ellipsoid therefore whose semi-axes are a, b, c, is entirely within the other, supposing them concentric; and A' therefore is within the ellipsoid, whose semi-axes are α, β, γ.

eliminating, therefore, n^2 and t^2, the denominator is

$$\begin{aligned} &l^2(a^2-\gamma^2)+m^2(\beta^2-\gamma^2)+\gamma^2 \\ &-2(aalr+b\beta ms+c\gamma n\sqrt{1-r^2-s^2}) \\ &+r^2(a^2-c^2)+s^2(b^2-c^2)+c^2. \end{aligned}$$

The second denominator differs from this only in the sign of

$$2c\gamma n\sqrt{1-r^2-s^2}.$$

Now, the attraction on A' parallel to γ, found in the same way

$$= \rho a\beta \times$$

$$\int_r\int_s\left(\frac{1}{\sqrt{(\overline{la-ar}|^2+\overline{mb-\beta s}|^2+\overline{nc-\gamma t}|^2)}} - \frac{1}{\sqrt{(\overline{la-ar}|^2+\overline{mb-\beta s}|^2+\overline{nc+\gamma t}|^2)}}\right).$$

The square of the first denominator, expanded as above,

$$\begin{aligned} &= l^2(a-c^2)+m^2(b^2-c^2)+c^2 \\ &-2(aalr+\beta bms+\gamma cn\sqrt{1-r^2-s^2}) \\ &+r^2(a^2-\gamma^2)+s^2(\beta^2-\gamma^2)+\gamma^2. \end{aligned}$$

But, by supposition,

$$a^2-c^2=a^2-\gamma^2, \quad b^2-c^2=\beta^2-\gamma^2;$$

hence, this is precisely equal to the square of the denominator of the first factor above: or the first fraction here = the first fraction above. Similarly, the second fraction here = the second fraction above. Hence, the whole expression under the sign of integration is the same in both. And the limits of integration are the same; therefore the integrals will be the same. In the first, the integral is multiplied by ρab, and in

the second, by $\rho\alpha\beta$: hence, the attraction of A parallel to c is to that of A' parallel to γ as ab to $\alpha\beta$.

44. In the same way, the attraction of A parallel to a, is to that of A' parallel to α, as bc to $\beta\gamma$: and the attraction of A parallel to b, is to that of A' parallel to β, as ac to $\alpha\gamma$. If the ellipsoid become a spheroid, by making $a=b$, then $\alpha=\beta$, and the forces parallel to c and γ, are as $a^2 : \alpha^2$: those parallel to a and α are as $ac : \alpha\gamma$: those parallel to b and β as ac to $\alpha\gamma$.

45. PROP. 21. To find the attraction of an oblate spheroid, whose ellipticity is small, on point without it.

Let c, the semi-axis of revolution of the spheroid, coincide with the axis of z: let f, g, h, be the co-ordinates of the attracted point. And, as in the last Proposition, let $f=l\alpha$, $g=m\alpha$, $h=n\gamma$, where

$$l^2+m^2+n^2=1, \text{ and } \alpha^2-\gamma^2=a^2-c^2.$$

$$\text{Let } a=c\,(1+e), \quad \alpha=\gamma\,(1+\epsilon):$$

then, rejecting e^2, &c., the last equation becomes

$$2\gamma^2\epsilon=2c^2e, \quad \text{or } \epsilon=\frac{c^2}{\gamma^2}e.$$

And from the equation

$$l^2+m^2+n^2=1,$$

$$\text{or } \frac{f^2+g^2}{\alpha^2}+\frac{h^2}{\gamma^2}=1, \text{ or } \frac{1}{\gamma^2}\{f^2+g^2+h^2-2\epsilon\,(f^2+g^2)\}=1,$$

$$\text{we get } \gamma^2=f^2+g^2+h^2-2\epsilon\,(f^2+g^2);$$

$$\text{but } \epsilon=\frac{c^2}{\gamma^2}e=\frac{c^2}{f^2+g^2+h^2}e;$$

$$\therefore\ \gamma^2=f^2+g^2+h^2-2e\,.\,\frac{c^2\,(f^2+g^2)}{f^2+g^2+h^2},$$

$$\text{and} \quad \gamma = \sqrt{f'^2+g'^2+h'^2} - e \cdot \frac{c^2\,(f'^2+g'^2)}{(f'^2+g'^2+h'^2)^{\frac{3}{2}}}.$$

Also, since $a^2 - \gamma^2 = a^2 - c^2 = 2\,c^2 e$,

$$\therefore \; a^2 = \gamma^2 + 2\,c^2 e = f'^2 + g'^2 + h'^2 + 2\,e\,\frac{c^2 h^2}{f'^2+g'^2+h'^2},$$

$$\therefore \; a = \sqrt{f'^2+g'^2+h'^2} + e\,\frac{c^2 h^2}{(f'^2+g'^2+h'^2)^{\frac{3}{2}}}.$$

46. Now, the co-ordinates of A' (Prop. 20.) $= la,\ mb,\ nc$: $= \dfrac{f'a}{a},\ \dfrac{gh}{a},\ \dfrac{hc}{\gamma}$. The distance of this point, therefore, from the axis of the spheroid

$$= \sqrt{\frac{f'^2 a^2}{a^2} + \frac{g'^2 a^2}{a^2}} = \frac{a}{a}\sqrt{f'^2+g'^2}.$$

And, as this point is within the spheroid whose semi-axes are a and γ, its attraction towards the axis, by (19),

$$= \frac{4\pi}{3}\rho\left(1 - \frac{2}{5}\epsilon\right) \cdot \frac{a}{a}\sqrt{f'^2+g'^2}:$$

hence, its attraction in the direction of x

$$= \frac{f'}{\sqrt{f'^2+g'^2}} \times \frac{4\pi}{3}\rho\left(1 - \frac{2}{5}\epsilon\right) \cdot \frac{a}{a}\sqrt{f'^2+g'^2}$$

$$= \frac{4\pi}{3}\rho \cdot \left(1 - \frac{2}{5}\epsilon\right) \cdot \frac{a}{a} \cdot f'.$$

Similarly, its attraction in the direction of y

$$= \frac{4\pi}{3}\rho \cdot \left(1 - \frac{2}{5}\epsilon\right) \cdot \frac{a}{a} \cdot g.$$

And, by (19), its attraction in the direction of z

$$= \frac{4\pi}{3}\rho\left(1 + \frac{4}{5}\epsilon\right) \cdot \frac{c}{\gamma} \cdot h.$$

47. By Prop. 20, therefore, the attraction on A

in the direction of $x = \frac{ac}{a\gamma} \times \frac{4\pi}{3}\rho\left(1 - \frac{2}{5}\epsilon\right) \cdot \frac{a}{a} \cdot f$

$$= \frac{4\pi}{3} \cdot \rho \cdot \frac{a^2 c}{a^2\gamma} \cdot \left(1 - \frac{2}{5}\epsilon\right) f;$$

that in the direction of $y = \frac{ac}{a\gamma} \times \frac{4\pi}{3}\rho\left(1 - \frac{2}{5}\epsilon\right) \cdot \frac{a}{a} \cdot g$

$$= \frac{4\pi}{3}\rho \cdot \frac{a^2 c}{a^2\gamma} \cdot \left(1 - \frac{2}{5}\epsilon\right) g;$$

that in the direction of $z = \frac{a^2}{a^2} \times \frac{4\pi}{3}\rho\left(1 + \frac{4}{5}\epsilon\right) \cdot \frac{c}{\gamma} \cdot h$

$$= \frac{4\pi}{3}\rho \cdot \frac{a^2 c}{a^2\gamma}\left(1 + \frac{4}{5}\epsilon\right) h.$$

Now $a^2 c = c^3(1+2e)$: $a^2\gamma$, from the expressions above,

$$= (f'^2 + g^2 + h^2)^{\frac{3}{2}} + e\,\frac{c^2(2h^2 - f'^2 - g^2)}{(f^2 + g^2 + h^2)^{\frac{1}{2}}};$$

$$\therefore \frac{a^2 c}{a^2\gamma} = \frac{c^3}{(f'^2 + g^2 + h^2)^{\frac{3}{2}}}\left\{1 + 2e - e\,\frac{c^2(2h^2 - f^2 - g^2)}{(f'^2 + g^2 + h^2)^2}\right\}.$$

$$\text{Also, } 1 - \frac{2}{5}\epsilon = 1 - e \cdot \frac{2}{5} \cdot \frac{c^2}{f'^2 + g^2 + h^2}$$

(putting $f'^2 + g^2 + h^2$ for γ^2).

$$\text{And } 1 + \frac{4}{5}\epsilon = 1 + e \cdot \frac{4}{5} \cdot \frac{c^2}{f'^2 + g^2 + h^2}.$$

Substituting these, we find, at length, the attraction of A in the direction of x

$$=\frac{4\pi}{3}\rho\cdot\frac{c^3}{(f^2+g^2+h^2)^{\frac{3}{2}}}\cdot\left\{1+2e-e\frac{c^2(12h^2-3f^2-3g^2)}{5(f^2+g^2+h^2)^2}\right\}f;$$

that in the direction of y

$$=\frac{4\pi}{3}\rho\cdot\frac{c^3}{(f^2+g^2+h^2)^{\frac{3}{2}}}\cdot\left\{1+2e-e\frac{c^2(12h^2-3f^2-3g^2)}{5(f^2+g^2+h^2)^2}\right\}g;$$

that in the direction of z

$$=\frac{4\pi}{3}\rho\cdot\frac{c^3}{(f^2+g^2+h^2)^{\frac{3}{2}}}\cdot\left\{1+2e-e\frac{c^2(6h^2-9f^2-9g^2)}{5(f^2+g^2+h^2)^2}\right\}h.$$

48. Prop. 22. To find the attraction of an oblate spheroid on a point without it; the spheroid being heterogeneous; and all the surfaces passing through points at which the density is the same, being spheroidal, of variable ellipticity.

Let EF, (fig. 15.) be the spheroid; let $E'F'$ be a spheroidal surface, at every one of whose points the density is the same, and $E''F''$ a spheroidal surface very near the former, of different ellipticity, at all of whose points the density is the same, but differing from that at the surface $E'F'$. Let $CF'=c$, $CF''=c+\delta c$. Since the ellipticity varies when the semi-axis of the spheroid is varied, e must be a function of c. Now the density of all the matter included between $E'F'$ and $E''F''$ is not uniform; but by diminishing δc, it may be made to approximate as nearly as we please to uniformity. Conceive, now, for the moment, the interior matter of the spheroid $E'F'$ to be of the same density as that at its surface, or to be equal to ρ; let its attraction in the direction of $x=\rho\,.\,A$. Then the attraction of the spheroid $E''F''$ in the same direction, will be the value which A receives when $c+\delta c$ is put for c, and when, instead of e we put the value of the ellipticity in the spheroid $E''F''$. But if we consider e as a function of c, this is included in considering the variation which it receives in consequence of the variation of c.

Hence A will be changed to $A+\frac{dA}{dc}\,\delta c+$ &c.; and therefore the attraction of the spheroid $E''F''$

$$=\rho\left(A+\frac{dA}{dc}\,\delta c+\&c.\right).$$

The difference of the attraction of the two spheroids, or the attraction of the shell included between them, is therefore ultimately $=\rho\,.\,\frac{dA}{dc}\,\delta c$. If, then, u be the attraction of the heterogeneous spheroid, whose polar semi-axis $=c$, we find $\frac{du}{dc}$ $=$ ultimate value of $\frac{\delta u}{\delta c}=\rho\,.\,\frac{dA}{dc}$;

$$\therefore\ u=\int_c \rho\,.\,\frac{dA}{dc};$$

where in the differentiation e must be considered as a function of c; and in the integration, ρ and e must both be considered as functions of c.

49. Now,

$$A=\frac{4\pi}{3}\left\{\frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}}\,c^3\,(1+2e)-\frac{12h^2-3f^2-3g^2}{5\,(f^2+g^2+h^2)^{\frac{7}{2}}}\,c^5e\right\}f;$$

and when we differentiate this with respect to c, since f, g, and h, are perfectly independent of c, the only variable terms will be $e^3\,(1+2e)$ and c^5e. Let

$$\int_c \rho\,.\,\frac{d\,(c^3.\,\overline{1+2e})}{de}=\phi\,(c);\quad \int_c \rho\,.\,\frac{d\,(c^5e)}{dc}=\psi\,(c);$$

both integrals being made to vanish when $c=0$. Then by the expression found in the last article, the attraction in the direction of x

$$= \frac{4\pi}{3} \cdot \left\{ \frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}} \cdot \phi(c) - \frac{12h^2 - 3f^2 - 3g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}} \psi(c) \right\} f.$$

Similarly from the values in (47), we find the attraction in the direction of y

$$= \frac{4\pi}{3} \left\{ \frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}} \cdot \phi(c) - \frac{12h^2 - 3f^2 - 3g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}} \psi(c) \right\} \cdot g.$$

And the attraction in the direction of z

$$= \frac{4\pi}{3} \left\{ \frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}} \cdot \phi(c) - \frac{6h^2 - 9f^2 - 9g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}} \psi(c) \right\} \cdot h.$$

It will be necessary to remark, that the second term in each expression is small, when the ellipticity is small, but the first is not small.

50. Prop. 23. To find the attraction of a heterogeneous shell on a point within it; the points of equal density being situated in spheroidal surfaces, and the interior and exterior surfaces being spheroids, in every part of which the density is the same.

Let f, g, h be the co-ordinates of the attracted point A', (fig. 15). Take $E'F'$, $E''F''$ as in the last Proposition, and suppose the whole spheroid $E'CF'$ to have the uniform density ρ. Then by (19), the attraction of the spheroid $E'CF'$ on the point A', directed towards the axis,

$$= \frac{4\pi}{3} \rho \sqrt{f^2+g^2} \cdot \left(1 - \frac{2}{5} e\right);$$

whence the attraction in the direction of x

$$= \frac{4\pi}{3} \rho \cdot f \left(1 - \frac{2}{5} e\right);$$

that in the direction of y

$$= \frac{4\pi}{3} \rho \cdot g \left(1 - \frac{2}{5} e\right).$$

And by (19), the attraction in the direction of z

$$= \frac{4\pi}{3} \rho \cdot h \left(1 + \frac{4}{5} e\right).$$

Then by exactly the same reasoning as that in Prop. 22, it may be shewn, that the attraction of the heterogeneous shell will be found by differentiating these expressions, (after taking away ρ) with respect to c, multiplying then by ρ, and integrating with regard to c. Let $\int_c \rho \cdot \frac{de}{dc} = \chi(c)$, the integral being made to vanish when $c = 0$. If we suppose the semi-polar axis of the interior surface to be c, that of the exterior to be c, then $\int_c \rho \frac{de}{dc}$ for the shell, between these limits, $= \chi(\mathrm{c}) - \chi(c)$. Hence it will easily be seen, that the force in the direction of x

$$= -\frac{4\pi}{3} \cdot \frac{2}{5} \{\chi(\mathrm{c}) - \chi(c)\} \cdot f,$$

that in the direction of y

$$= -\frac{4\pi}{3} \cdot \frac{2}{5} \{\chi(\mathrm{c}) - \chi(c)\} \cdot g,$$

that in the direction of z

$$= \frac{4\pi}{3} \cdot \frac{4}{5} \{\chi(\mathrm{c}) - \chi(c)\} \cdot h.$$

51. Prop. 24. To find the attraction of a heterogeneous spheroid of the same kind as that in Prop. 22., upon any point within it.

Let γ be the polar semi-axis of that spheroidal surface which passes through the given point, and through all other points at which the density is the same. And let c be the semi-axis of the exterior surface. Then the given point is external to all the spheroids whose semi-axes are less than γ, and to these, therefore, the integration in Prop. 22. must be applied. Taking these integrals, then, from $c=0$ to $c=\gamma$, we find for the forces in the direction of x, y and z,

$$\frac{4\pi}{3}\left\{\frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}}\,\phi(\gamma)-\frac{12h^2-3f^2-3g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}}\,\psi(\gamma)\right\}f,$$

$$\frac{4\pi}{3}\left\{\frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}}\,\phi(\gamma)-\frac{12h^2-3f^2-3g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}}\,\psi(\gamma)\right\}g.$$

$$\frac{4\pi}{3}\left\{\frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}}\,\phi(\gamma)-\frac{6h^2-9f^2-9g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}}\,\psi(\gamma)\right\}h.$$

Again, the given point is interior to all the spheroidal surfaces whose polar semi-axes are greater than γ, and less than c; and, therefore, the expressions of Prop. 23. must be taken between these limits. These give us for the forces in the directions of x, y and z,

$$-\frac{4\pi}{3}\cdot\frac{2}{5}\{\chi(c)-\chi(\gamma)\}f$$

$$-\frac{4\pi}{3}\cdot\frac{2}{5}\{\chi(c)-\chi(\gamma)\}g,$$

$$\frac{4\pi}{3}\cdot\frac{4}{5}\{\chi(c)-\chi(\gamma)\}h.$$

and if these be added to those above, we shall have the whole forces arising from the attraction of the spheroid.

52. Besides these, if the spheroid revolve round its polar axis in the time T, every point will experience a centrifugal

force proportional to its distance from the axis. Resolving this in the directions of x and y, we have for the parts in those directions $\frac{4\pi^2}{T^2}.f$ and $\frac{4\pi^2}{T^2}.g$, which must be subtracted from the attractions. Collecting all the terms, the force in the direction of x =

$$\frac{4\pi}{3}\left\{\frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}}\phi(\gamma) - \frac{12h^2-3f^2-3g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}}\psi(\gamma)\right.$$
$$\left.\frac{2}{5}\{\chi(c)-\chi(\gamma)\} - \frac{3\pi}{T^2}\right\}f = F.$$

That in the direction of y =

$$\frac{4\pi}{3}\left\{\frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}}\phi(\gamma) - \frac{12h^2-3f^2-3g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}}\psi(\gamma)\right.$$
$$\left.-\frac{2}{5}\{\chi(c)-\chi(\gamma)\} \quad \frac{3\pi}{T^2}\right\}g = G.$$

That in the direction of z =

$$\frac{4\pi}{3}\left\{\frac{1}{(f^2+g^2+h^2)^{\frac{3}{2}}}\phi(\gamma) - \frac{6h^2-9f^2-9g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}}\psi(\gamma)\right.$$
$$\left.+\frac{4}{5}\{\chi(c)-\chi(\gamma)\}\right\}h \ldots\ldots = H.$$

53. **Prop. 25.** **To find the ellipticities of the spheroids of equal density, that the force at any point may be perpendicular to the surface of the spheroid, passing through that point.**

Let EF, (fig. 16.) be the surface of equal density, passing through the point P, whose co-ordinates CM, MN, NP, are f, g, h; let γ, and a or $\gamma(1+\epsilon)$, be the polar and equatoreal radii; FPG the generating ellipse, which passes through P; PQ a normal at P; QR perpendicular to MN. Then a force perpendicular to the surface at P, may be represented by PQ,

and may be resolved into three represented by PN, NR, RQ. Hence, if the whole force be in the direction PQ, the part in the direction of h will be to that in the direction of g, or H : G, as $PN : NR$. But $\frac{NQ}{CN} = \frac{a^2}{\gamma^2}$, by Conic Sections;

$$\frac{NR}{MN} = \frac{a^2}{\gamma^2}; \therefore NR = MN \cdot \frac{a^2}{\gamma^2} = g\frac{a^2}{\gamma^2}$$

$$\text{Hence, } H : G \quad h : g\frac{a^2}{\gamma^2};$$

$$\frac{H}{h} : \frac{G}{g} \quad \frac{\gamma^2}{a^2} \qquad + 2\epsilon$$

$$\therefore \frac{H}{h} = 1 + 2\epsilon \cdot \frac{G}{g}$$

Substituting, in this equation, the expressions found in (52), and multiplying 2ϵ into no term but the first, (as all the other terms are small when ϵ is small), we find

$$\frac{2\epsilon \cdot \phi(\gamma)}{(f^2+g^2+h^2)^{\frac{3}{2}}} - \frac{6\psi(\gamma)}{5(f^2+g^2+h^2)^{\frac{5}{2}}} - \frac{3}{5}\{\chi(c) - \chi(\gamma)\} - \frac{3\pi}{T^2} = 0.$$

But $f^2 + g^2 + h^2$ differs from γ^2 only by a quantity which depends upon ϵ; putting therefore γ^2 for $f^2 + g^2 + h^2$, since all the terms are small,

$$\frac{\epsilon \cdot \phi(\gamma)}{\gamma^3} - \frac{3}{5} \cdot \frac{\psi(\gamma)}{\gamma^5} - \frac{3}{5}\{\chi(c) - \chi(\gamma)\} - \frac{3\pi}{2T^2} = 0.$$

Or, since this must be true, whatever be the value of γ, we may put for γ the general letter c, and we have

$$\frac{e \cdot \phi(c)}{c^3} - \frac{3}{5} \cdot \frac{\psi(c)}{c^5} - \frac{3}{5}\{\chi(c) - \chi(c)\} - \frac{3\pi}{2T^2} = 0.$$

54. If we put for $\phi(c)$, $\psi(c)$ and $\chi(c)$ their values

$$\int_c \rho \cdot \frac{d\,(c^3 . \overline{1 + 2e})}{dc} \text{ or } \int_c . \rho . \frac{d\,(c^3)}{dc}, \text{ nearly,}$$

$$\int_c \rho \cdot \frac{d\,(c^5 e)}{dc}, \text{ and } \int_c \rho \frac{de}{dc}, \text{ (49 and 50),}$$

and differentiate,

$$\frac{d^2 e}{dc^2} + \frac{2\rho c^2}{\int_c \rho c^2} \cdot \frac{de}{dc} + \left(\frac{2\rho c}{\int_c \rho c^2} - \frac{6}{c^2}\right) e = 0.$$

This differential equation it is always possible to integrate, at least by series, when ρ is given in terms of c; and the two arbitrary constants will enable us to make the value of e satisfy the equation from which it is derived. Hence, when ρ is given in terms of c, it is always possible to find the ellipticity of every surface of equal density, so as to satisfy the condition of this Proposition.

55. Prop. 26. When this condition is satisfied, the fluid is in equilibrium.

To shew this, we shall suppose any number of canals terminated in the external surface, to be drawn to any point within, and we shall find the pressure produced by the fluid in one of these canals, on the given point; we shall then shew, from the expression, that this is the same for every canal, in whatever direction it be drawn.

Let P, (fig. 17), be any point in one canal, p very near it; let $RP = s$, $Rp = s + \delta s$; let the co-ordinates of P be f, g, h; those of p, $f + \delta f$, $g + \delta g$, $h + \delta h$. Then if a parallelopiped be constructed of which Pp or δs is the diagonal, (since it is ultimately a straight line), and whose sides are parallel to f, g, h, respectively; the lengths of those sides will be δf, δg, δh. To find how much the pressure of the fluid at P exceeds that at p, we must resolve each of the forces acting on its particles into two, one in the direction of pP, and the other perpendicular to pP; then we must add together the former, and neglect the

latter. The forces are F, G, H in the directions of f, g, and h; hence, the sum of the forces in the direction of pP, is

$$F \cos pPq + G \cos pPr + H \cos pPt,$$

$$= F\frac{\delta f}{\delta s} + G\frac{\delta g}{\delta s} + H\frac{\delta h}{\delta s}.$$

And the matter, upon which they act, is $\rho . \delta s$, (the section of the canal being supposed $= 1$); therefore the pressure which they produce $=$ ultimately

$$\rho . (F\delta f + G\delta g + H\delta h).$$

If, then, we can find a quantity V, such that

$$\frac{dV}{df} = -\rho F, \frac{dV}{dg} = -\rho G, \frac{dV}{dh} = -\rho H,$$

V, taken between the proper limits, will be the pressure.

56. Now, taking the expressions in (52), and integrating them by parts,

$$-V = \frac{4\pi}{3}\left\{\frac{-\rho . \phi(\gamma)}{(f^2+g^2+h^2)^{\frac{1}{2}}} + \rho . \psi(\gamma) . \frac{2h^2-f^2-g^2}{5(f^2+g^2+h^2)^{\frac{5}{2}}}\right.$$

$$\left.+\frac{1}{5}\rho\{\chi(c)-\chi(\chi)\} . (2h^2-f^2-g^2) - \frac{3\pi}{2T^2}\rho(f^2+g^2)\right\}.$$

$$+\frac{4\pi}{3}\int_\gamma . \left\{\frac{1}{(f^2+g^2+h^2)^{\frac{1}{2}}} . \frac{d.\rho\phi(\gamma)}{d\gamma} - \frac{2h^2-f^2-g^2}{5(f^2+g^2+h^2)^{\frac{5}{2}}} . \frac{d.\rho\psi(\gamma)}{d\gamma}\right.$$

$$\left.- \frac{2h^2-f^2-g^2}{5} . \frac{d . \rho\{\chi(c) - \chi(\gamma)\}}{d\gamma} + \frac{3\pi}{2T^2} . \overline{f^2+g^2} . \frac{d\rho}{d\gamma}\right\},$$

where γ is the polar radius of the spheroid of equal density, passing through the point whose co-ordinates are f, g, h. The equatoreal semi-axis being $\gamma(1+\epsilon)$, we have

$$\frac{f'^2+g^2}{\gamma^2(1+2\epsilon)}+\frac{h^2}{\gamma^2}=1,$$

$$\text{whence, } f'^2+g^2+h^2=\gamma^2+2\epsilon\,.\,f'^2+g^2.$$

Substituting this expression in the first term of each line, which is large, and putting γ^2 only for $f^2+g^2+h^2$ in the other terms, which are small, and observing, that then

$$2h^2-f'^2-g^2=2\gamma^2-3\,.\,f'^2+g^2,$$

we have, for the first line,

$$\frac{4\pi}{3}\rho\left\{-\frac{\phi(\gamma)}{\gamma}+\frac{\epsilon\,.\,\phi(\gamma)}{\gamma^3}(f'^2+g^2)+\frac{2}{5}\cdot\frac{\psi(\gamma)}{\gamma^3}\right.$$

$$-\frac{3}{5}\cdot\frac{\psi(\gamma)}{\gamma^5}(f'^2+g^2)+\frac{2}{5}\gamma^2\{\chi(c)-\chi(\gamma)\}$$

$$\left.-\frac{3}{5}\{\chi(c)-\chi(\gamma)\}(f'^2+g^2)-\frac{3\pi}{2T^2}(f'^2+g^2)\right\}.$$

But, by Prop. 25,

$$\frac{\epsilon\,.\,\phi(\gamma)}{\gamma^3}-\frac{3}{5}\cdot\frac{\psi(\gamma)}{\gamma^5}-\frac{3}{5}\{\chi(c)-\chi(\gamma)\}-\frac{3\pi}{2T^2}=0,$$

and the first line reduces itself to this expression:

$$\frac{4\pi}{3}\rho\left\{-\frac{\phi(\gamma)}{\gamma}+\frac{2}{5}\cdot\frac{\psi(\gamma)}{\gamma^3}+\frac{2}{5}\gamma^2\{\chi(c)-\chi(\gamma)\}\right\}\ \ldots.\ (A).$$

Making the same substitutions in the second line, it becomes

$$\frac{4\pi}{3}\int_\gamma\left\{\frac{1}{\gamma}\cdot\frac{d\,.\,\rho\phi(\gamma)}{d\gamma}-\frac{2}{5\gamma^3}\cdot\frac{d\,.\,\rho\psi(\gamma)}{d\gamma}\right.$$

$$-\frac{2\gamma^2}{5}\cdot\frac{d\,.\,\rho\,\{\chi(c)-\chi(\gamma)\}}{d\gamma}+\overline{f^2+g^2}\left(-\frac{\epsilon}{\gamma^3}\cdot\frac{d\,.\,\rho\phi(\gamma)}{d\gamma}\right.$$

$$\frac{3}{5\gamma^5}\cdot\frac{d\,.\,\rho\psi(\gamma)}{d\gamma}+\frac{3}{5}\cdot\frac{d\,.\,\rho\,\{\chi(c)-\chi(\gamma)\}}{d\gamma}+\frac{3\pi}{2T^2}\,\frac{d\rho}{d\gamma}$$

The coefficient of f^2+g^2 is

$$\frac{d\rho}{d\gamma}\cdot\left\{-\frac{\epsilon}{\gamma^3}\cdot\phi(\gamma)+\frac{3}{5\gamma^5}\psi(\gamma)+\frac{3}{5}\{\chi(c)-\chi(\gamma)\}+\frac{3\pi}{2T^2}\right\}$$

$$+\rho\cdot\left\{-\frac{\epsilon}{\gamma^3}\cdot\frac{d\cdot\phi(\gamma)}{d\gamma}+\frac{3}{5\gamma^5}\cdot\frac{d\cdot\psi(\gamma)}{d\gamma}-\frac{3}{5}\cdot\frac{d\cdot\chi(\gamma)}{d\gamma}\right\},$$

of which the first part is equal to 0, by the equation of Prop. 25, and the second part is identically $=0$, as will be seen upon putting for $\frac{d\cdot\phi(\gamma)}{d\gamma}$, $\frac{d\cdot\psi(\gamma)}{d\gamma}$, and $\frac{d\cdot\chi(\gamma)}{d\gamma}$ their values

$$\frac{d\cdot\gamma}{d\gamma}\qquad\frac{d\cdot\gamma^5\epsilon}{d\gamma}.\qquad\text{and }\rho\frac{d\epsilon}{d\gamma}$$

Hence, the second line reduces itself to this,

$$\frac{4\pi}{3}\int_\gamma\left\{\frac{1}{\gamma}\cdot\frac{d.\rho\phi(\gamma)}{d\gamma}-\frac{2}{5\gamma^3}\cdot\frac{d.\rho\psi(\gamma)}{d\gamma}-\frac{2\gamma^2}{5}\cdot\frac{d.\rho\{\chi(c)-\chi(\gamma)\}}{d\gamma}\right\}\ (B).$$

The expression, therefore, for the pressure, or $-(A+B)$, is a function of γ only. Let it $=\Phi(\gamma)$; observing that the pressure at the surface is 0, the pressure any where within will be $\Phi(\gamma)-\Phi(c)$.

57. Now this expression does not at all depend on the form of the canal, nor upon the position of its extremities, provided it be terminated in the surface; and, consequently, the pressure at the point R is the same from the fluid in the canal SR, as from that in TR, or any other canal from R, terminated in the external surface; the point R, therefore, being equally pressed on all sides, will be at rest. And the same may be proved for every other point, therefore every point will be at rest.

58. It will be observed, that the expression for the pressure at any point, does not depend upon the co-ordinates of that point, but upon the polar semi-axis of the surface of equal density, passing through that point; and, therefore, the pressure is the same, where the density is the same. The equation of equilibrium, then, is the same, whether the density be supposed to depend on the pressure, or not.

59. Prop. 27. To find the whole force on any point at the external surface.

The forces on any point in the directions of x, y, z, being F, G, and H, the whole force will be $\sqrt{F'^2+G^2+H^2}$. The forces on a point at the surface will be found by putting c for γ, in the expressions of (52). Thus we find for a point at the surface,

$$F=\frac{4\pi}{3}\left\{\frac{1}{(f'^2+g^2+h^2)^{\frac{3}{2}}}\cdot\phi(\mathrm{c})-\frac{12h^2-3f'^2-3g^2}{5(f'^2+g^2+h^2)^{\frac{7}{2}}}\psi(\mathrm{c})-\frac{3\pi}{T'^2}\right\}.f,$$

$$G=\frac{4\pi}{3}\left\{\frac{1}{(f'^2+g^2+h^2)^{\frac{3}{2}}}\cdot\phi(\mathrm{c})-\frac{12h^2-3f'^2-3g^2}{5(f'^2+g^2+h^2)^{\frac{7}{2}}}\psi(\mathrm{c})-\frac{3\pi}{T'^2}\right\}g,$$

$$H=\frac{4\pi}{3}\left\{\frac{1}{(f'^2+g^2+h^2)^{\frac{3}{2}}}\phi(\mathrm{c})-\frac{6h^2-9f'^2-9g^2}{5(f'^2+g^2+h^2)^{\frac{7}{2}}}\psi(\mathrm{c})\right\}h,$$

and hence $\sqrt{F'^2+G^2+H^2}$, (observing that the first term only of each expression is large,)

$$=\frac{4\pi}{3}\left\{\frac{\phi(\mathrm{c})}{f'^2+g^2+h^2}-\frac{6h^2-3.\overline{f'^2+g^2}}{5(f'^2+g^2+h^2)^{3}}\psi(\mathrm{c})-\frac{3\pi}{T^2}\cdot\frac{f'^2+g^2}{\sqrt{f'^2+g^2+h^2}}\right\}$$

If e be the ellipticity of the external surface,

$$f'^2+g^2+h^2=\mathrm{c}^2+2\mathrm{e}.\overline{f'^2+g^2},$$

which gives for the force

$$\frac{4\pi}{3}\left\{\frac{\phi(\mathrm{c})}{\mathrm{c}^2}-\frac{2\mathrm{e}.\phi(\mathrm{c})}{\mathrm{c}^4}.\overline{f'^2+g^2}-\frac{6\mathrm{c}^2-9.\overline{f'^2+g^2}}{5\,\mathrm{c}^6}\psi(\mathrm{c})-\frac{3\pi}{\mathrm{c}T^2}.\overline{f'^2+g^2}\right\},$$

60. Suppose at the equator the centrifugal force $= m \times$ gravity, m being small $\left(\text{in the earth it is } \frac{1}{289}\right)$. The gravity

at the equator, neglecting the small terms, is $\frac{4\pi}{3}\cdot\frac{\phi(c)}{c^2}$: the centrifugal force is $\frac{4\pi^2}{T'^2}\cdot a = \frac{4\pi^2}{T'^2}c$, nearly. Hence,

$$\frac{4\pi^2}{T'^2}c = m\frac{4\pi}{3}\cdot\frac{\phi(c)}{c^2}; \quad \therefore \quad \frac{3\pi}{cT'^2} = m\frac{\phi(c)}{c^4}.$$

Hence, the whole force at the surface

$$= \frac{4\pi}{3}\left\{\frac{\phi(c)}{c^2} - \frac{6}{5}\cdot\frac{\psi(c)}{c^4} - \left(\overline{2e+m}\cdot\frac{\phi(c)}{c^4} - \frac{9}{5}\cdot\frac{\psi(c)}{c^6}\right)\overline{f'^2+g^2}\right\}.$$

61. The equation of Prop. 25, becomes, at the surface,

$$\frac{e\cdot\phi(c)}{c^3} - \frac{3}{5}\cdot\frac{\psi(c)}{c^5} - \frac{3\pi}{2T'^2} = 0,$$

$$\text{or, since } \frac{3\pi}{2T'^2} = \frac{m}{2}\cdot\frac{\phi(c)}{c^3}.$$

$$\left(e - \frac{m}{2}\right)\frac{\phi(c)}{c^3} - \frac{3}{5}\cdot\frac{\psi(c)}{c^5} = 0.$$

Substituting from this the value of $\psi(c)$, we find, for the force at any point of the surface, the following simple expression

$$\frac{4\pi}{3}\cdot\frac{\phi(c)}{c^2}\cdot\left\{1 - \overline{2e-m} - \frac{5m}{2} - e\cdot\frac{f'^2+g^2}{c^2}\right\}.$$

62. For the force at the pole we must make $f = 0$, $g = b$, and the force, therefore,

$$= \frac{4\pi}{3}\cdot\frac{\phi(c)}{c^2}(1 - \overline{2e-m}).$$

For the force at the equator we must make $f'^2 + g^2 = a^2 = c^2$, nearly, and, therefore, the equatoreal gravity

$$= \frac{4\pi}{3}\frac{\phi(c)}{c^2}\left\{1 - \overline{2e-m} - \frac{5m}{2} - e\right\}.$$

The excess of the former above this

$$= \frac{4\pi}{} \cdot \frac{\phi(c)}{} \frac{5m}{} - e:$$

and the ratio of this excess to the equatoreal gravity is $\frac{5m}{2} - e$.

Let this $= n$: then, $n + e = \frac{5m}{2}$: a very remarkable proposition, which may be thus stated: "Whatever be the law of the Earth's density, if the ellipticity of the surface be added to the ratio which the excess of the polar above the equatoreal gravity bears to the equatoreal gravity, their sum will be $\frac{5m}{2}$, m being the ratio of the centrifugal force at the equator to the equatoreal gravity." This is called *Clairaut's Theorem*.

63. Prop. 28. To find an expression for gravity at any point of the surface, in terms of the latitude.

Suppose EF, fig. 16, to represent the Earth's surface, PQ a normal at P. Then, PQN is the latitude of P, and

$$QN = PQ \cdot \cos l.$$

Now, as we shall have to substitute only in the small terms of the equation, $PQ = c$ nearly, and $QN = CN$ nearly $= \sqrt{f'^2 + g^2}$; hence $\sqrt{f'^2 + g^2} = c \cdot \cos l$ nearly. Substituting this in the expression of (61), gravity

$$= \frac{4\pi}{3} \frac{\phi(c)}{c^2} \left\{1 - \overline{2e - m - \overline{\frac{5m}{2} - e}} \cdot \cos^2 l\right\},$$

$$\text{or } = \frac{4\pi}{3} \cdot \frac{\phi(c)}{c^2} \cdot \left\{1 - e + \frac{3m}{2} + \frac{5m}{2} - e \cdot \sin^2 l\right\},$$

$$\text{or } = \frac{4\pi}{3} \cdot \frac{\phi(c)}{c^2} \cdot \left(1 - \overline{e + \frac{3m}{2}}\right)\left(1 + \overline{\frac{5m}{2} - e} \cdot \sin^2 l\right).$$

Gravity, therefore, may be generally expressed by the formula $E\,(1 + n\,.\sin^2 l)$, where E = equatoreal gravity, and

$$n + e = \frac{5\,m}{}$$

64. Prop. 29. To find the ellipticity of the Earth on any assumed law of density of the strata.

Differentiating the equation

$$\frac{e\,.\,\phi(c)}{c^2} - \frac{3}{5}\,.\,\frac{\psi(c)}{c^3} - \frac{3}{5}\,\{\chi(c) - \chi(c)\} - \frac{3\,\pi}{2\,T^2} = 0\,..\,(\mathrm{I}),$$

we find

$$c^3\,\phi(c)\,.\,\frac{de}{dc} + 3\,\psi(c) - 3\,e\,.\,c^2\,\phi(c) = 0\,....\,(\mathrm{II}),$$

and differentiating this,

$$\frac{d^2e}{dc^2} + \frac{2\rho c^2}{\int_c \rho c^2}\,.\,\frac{de}{dc} + \left(\frac{2\,\rho c}{\int_c \rho c^2} - \frac{6}{c^2}\right) e = 0\,....\,(\mathrm{III}).$$

Now, when ρ is given in terms of c, we must substitute it in this last equation, and by integration find e. The expression will contain two arbitrary constants: one of these may, in general, be conveniently determined by substituting in equation (II), and the other by substituting in equation (I). Equation (III) may be transformed into one of a simpler form, thus.

Let $\int_c \rho c^2 = p$, and let $pe = v$, or $e = \dfrac{v}{p}$: then, upon substituting in (III), we get

$$\frac{d^2v}{dc^2} \quad \frac{6\,v}{c^2} \quad \cdot v\,\frac{c}{p}\,.\,\frac{d\rho}{dc} = 0.$$

65. *Example.* Suppose $\rho = A\,.\,\dfrac{\sin q c}{c}$, A and q being constant. As this gives a density diminishing from the center

to the surface, it is probable that it will pretty nearly represent that of the Earth. On substituting in (IV), we get

$$\frac{d^2 v}{dc^2} - \frac{6v}{c^2} + q^2 v = 0:$$

the complete solution of which is

$$v = C\left(\sin \overline{qc + C'} + \frac{3}{qc} \cos \overline{qc + C'} - \frac{3}{q^2 c^2} \sin \overline{qc + C'}\right),$$

C and C' being arbitrary constants. Observing that $\phi(c) = 3p$,

$$\text{and } \psi(c) = \int_c \rho \cdot \frac{d}{dc}\left(\frac{c^5 v}{p}\right) = \frac{\rho c^5 v}{p} - \int_c \frac{c^5 v}{p} \cdot \frac{d\rho}{dc},$$

(the integral being made to vanish when $c = 0$), and, after finding these values, substituting then in equation (II), we find that it reduces itself to

$$45\, C \cdot \sin C' = 0.$$

C', therefore, must $= 0$; and, therefore,

$$v = C\left(\sin qc + \frac{3}{qc} \cos qc - \frac{3}{q^2 c^2} \sin qc\right).$$

Now, making use of this value in equation (I), and observing that

$$\chi(c) = \int_c \rho \frac{d}{dc}\left(\frac{v}{p}\right) = \frac{\rho v}{p} - \int_c \frac{v}{p} \cdot \frac{d\rho}{dc};$$

and putting for $\frac{3\pi}{2T^2}$ the value $\frac{m}{2} \cdot \frac{\phi(c)}{c^3}$ found in (60), the equation becomes

$$-A \cdot \frac{3m}{2}\left(\frac{\sin qc}{q^2 c^3} - \frac{1}{qc^2} \cos qc\right)$$

$$-\frac{3}{5}\,C\sin qc\,.\,\frac{\dfrac{\sin qc}{c}+\dfrac{3\cos qc}{qc^2}-\dfrac{3\sin qc}{q^2c^3}}{\dfrac{\sin qc}{q^2}-\dfrac{c\cos qc}{q}}$$

$$+\frac{3}{5}\,C\left(\frac{q\,.\cos qc}{c^2}-\frac{\sin qc}{c^3}\right)=0.$$

Determining from this equation the value of $\frac{C}{A}$, we find the ellipticity of any stratum, whose polar semi-axis is c

$$5\,m\;\frac{\left(1-\dfrac{qc}{\tan qc}\right)}{2-q^2c^2-\dfrac{qc}{\tan qc}-\dfrac{q^2c^2}{\tan^2 qc}}\;\frac{1-\dfrac{3}{q^2c^2}+\dfrac{3}{qc\,.\tan qc}}{1-\dfrac{qc}{\tan qc}}$$

For the ellipticity of the surface we must make $c = \mathrm{c}$: thus we obtain

$$\mathrm{e}=\frac{5\,m}{2}\,.\,\frac{\left(1-\dfrac{qc}{\tan qc}\right).\left(1-\dfrac{3}{q^2c^2}+\dfrac{3}{qc\,.\tan qc}\right)}{2-q^2c^2-\dfrac{qc}{\tan qc}-\dfrac{q^2c^2}{\tan^2 qc}}$$

If $z=1-\dfrac{qc}{\tan qc}$, $\quad \mathrm{e}=\dfrac{5\,m}{2}\,.\,\dfrac{1-\dfrac{3z}{q^2c^2}}{3-z-q^2c^2}$

66. Suppose $q=\frac{5}{6}\cdot\frac{\pi}{c}$. Then $qc=\frac{5\pi}{6}$: $z=5{,}53452$; and $\mathrm{e}=\frac{5\,m}{2}\times 0{,}37703$. And, since n the increase of gravity at the pole $=\frac{5\,m}{2}-\mathrm{e}$, by (62), $n=\frac{5\,m}{2}\times{,}62297$. In the

Earth $m = \frac{1}{289} : \frac{5m}{2} = \frac{1}{115}$; therefore on these assumptions, $e = \frac{1}{305}$, $n = \frac{1}{184,6}$ These are very nearly the same as the observed values of e and n.

67.* PROP. 30. To compare the mean density with the density at the surface. If the whole spheroid consisted of matter whose density = mean density, its mass would be the same as it actually is. Hence, the mean density $= \frac{\text{mass}}{\text{volume}}$. Now, neglecting the ellipticity, the mass may be found by considering it as a series of spherical shells of different density: and since the surface of one of these, whose radius is c, is $4\pi . c^2$, the mass of the shell whose thickness is δc, is ultimately $4\pi . \rho c^2 \delta c$, and putting u for the mass,

$$\frac{du}{dc} = 4\pi\rho c^2, \quad u = 4\pi \int_c \rho c^2 = 4\pi . p,$$

and u the whole mass $= 4\pi p$. And the volume $= \frac{4\pi}{3} c^3$. Hence, the mean density $= \frac{3p}{c^3}$. And, if the density at the surface $= \rho'$, the ratio of the mean density to that at the surface $= \frac{3p}{c^3 . \rho'}$.

68. *Example.* Suppose the same law of density and the same value of q as before.

$$\text{Then } p = A\left(\frac{\sin qc}{q^3} - \frac{c}{q}\cos qc\right) : \rho = \frac{A \sin qc}{c};$$

$$\therefore \frac{3p}{c^3\rho'} = \frac{3}{c^2 . \sin qc} . \left(\frac{\sin qc}{q^3} - \frac{c}{q}\cos qc\right)$$

$$= \frac{3}{q^2c^2}\left(1 - \frac{qc}{\tan qc}\right) = \frac{3z}{q^2c^2} = 2,4225.$$

69. PROP. 31. To find the effect produced by the ellipticity of the Earth on the motion of the Moon.

Let the co-ordinates of the Moon, referred to the Earth's center, be f, g, h: h being perpendicular to the plane of the equator, and f in the intersection of the plane of the equator and ecliptic. Let NP, fig. 18, be the intersection of the plane of the ecliptic by the plane passing through g and h; and drawing MP perpendicular to NP, let $NP=k$, $PM=l$: then f, k, l, are the three co-ordinates of M, f and k being in the plane of the ecliptic. Let $EP=\rho$, tan $MEP=s$, $PEN=\theta$, PNO = inclination of ecliptic to equator $=\omega$. Then, if PQ, PR, be drawn parallel to h and g,

$$g=NQ-RP=k\cos\omega-l\sin\omega:$$

$$h=QP+MR=k\sin\omega+l\cos\omega.$$

But $k=\rho\sin\theta$; $l=\rho s$; also $f=\rho\cos\theta$. Thus we get for the original co-ordinates these values,

$$f=\rho\cos\theta,$$

$$g=\rho(\sin\theta\cos\omega-s\,.\sin\omega),$$

$$h=\rho(\sin\theta\sin\omega+s\,.\cos\omega).$$

Let F, G, H, be the forces in the directions of f, g, and h; K and L those in the directions of k and l; and P, T, S, the forces parallel to ρ, perpendicular to ρ in the plane of the ecliptic, and perpendicular to the ecliptic. Then

$$K=G\cos\omega+H\sin\omega,$$

$$L=H\cos\omega-G\sin\omega.$$

Also $$P=F\cos\theta+K\sin\theta;$$

$$T=F\sin\theta-K\cos\theta;$$

$$S=L.$$

Substituting the values of K and L,

$$P = F\cos\theta + (G\cos\omega + H\sin\omega)\sin\theta,$$

$$T = F\sin\theta - (G\cos\omega + H\sin\omega)\cos\theta,$$

$$S = H\cos\omega - G\sin\omega.$$

70. Now, by Prop. 22,

$$F = \frac{4\pi}{3}\left\{\frac{\phi(c)}{(f^2+g^2+h^2)^{\frac{3}{2}}} - \frac{12h^2 - 3f^2 - 3g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}}\,\psi(c)\right\}.f,$$

$$G = \frac{4\pi}{3}\left\{\frac{\phi(c)}{(f^2+g^2+h^2)^{\frac{3}{2}}} - \frac{12h^2 - 3f^2 - 3g^2}{(f^2+g^2+h^2)^{\frac{7}{2}}}\,\psi(c)\right\}.g,$$

$$H = \frac{4\pi}{3}\left\{\frac{\phi(c)}{(f^2+g^2+h^2)^{\frac{3}{2}}} - \frac{6h^2 - 9f^2 - 9g^2}{5(f^2+g^2+h^2)^{\frac{7}{2}}}\,\psi(c)\right\}h.$$

Upon substituting these values in the expressions for P, T, and S, the quantity multiplied by $\psi(c)$ in each is rather complicated, and it is, therefore, proper to take only those terms which, when substituted in the equations in Art. 41. of the Physical Astronomy, will be much increased by integration. With this restriction, we have

$$P = \frac{4\pi}{3}\cdot\frac{\phi(c)}{\rho^2(1+s^2)^{\frac{3}{2}}},$$

$$T = \frac{4\pi}{3}\cdot\frac{6\psi(c)}{5\rho^4}\sin\omega\,.\cos\omega\,.\cos\theta\,.\,s,$$

$$S = \frac{4\pi}{3}\cdot\left\{\frac{\phi(c)\,.\,s}{\rho^2(1+s^2)^{\frac{3}{2}}} + \frac{6\psi(c)}{5\rho^4}\,.\sin\omega\,.\cos\omega\,.\sin\theta\right\}$$

Let E be the mass of the Earth. The volume of a spheroid, whose semi-axes are c and $c(1+e)$, is $\frac{4\pi}{3}c^3(1+2e)$: hence, the volume included between two spheroidal surfaces, is ultimately

$$\frac{4\pi}{3} \cdot \frac{d(c^3 \cdot \overline{1+2e})}{dc} \delta c:$$

and if the density be ρ, the increment of the mass is

$$\frac{4\pi}{3} \cdot \rho \cdot \frac{d(c^3 \cdot \overline{1+2e})}{dc} \delta c;$$

therefore the mass

$$= \frac{4\pi}{3} \int_c \rho \frac{d(c^3 \cdot \overline{1+2e})}{dc} = \frac{4\pi}{3} \phi(c); \quad \therefore \; E = \frac{4\pi}{3} \phi(c).$$

Also, by the equation of (61),

$$\left(e - \frac{m}{2}\right) \cdot \frac{\phi(c)}{c^3} - \frac{3}{5} \cdot \frac{\psi(c)}{c^5} = 0,$$

whence

$$\frac{4\pi}{3} \cdot \frac{6}{5} \cdot \psi(c) = \left(e - \frac{m}{2}\right) \cdot 2c^2 \cdot \phi(c) \cdot \frac{4\pi}{3} = \left(e - \frac{m}{2}\right) 2c^2 \cdot E.$$

Hence, we finally obtain

$$P = \frac{E}{\rho^2 (1+s^2)^{\frac{3}{2}}} - \frac{E \cdot u^2}{(1+s^2)^{\frac{3}{2}}},$$

$$T = -\left(e - \frac{m}{2}\right) \frac{2c^2}{\rho^4} \cdot E \cdot \sin\omega \cdot \cos\theta \cdot s$$

$$= -\left(e - \frac{m}{2}\right) \cdot 2c^2 u^4 E \cdot \sin\omega \cdot \cos\omega \cdot \cos\theta \cdot s,$$

$$S = \frac{Es}{\rho^2 (1+s^2)^{\frac{3}{2}}} + e - \frac{m}{2} \cdot \frac{2c^2}{\rho^4} E \cdot \sin\omega \cdot \cos\omega \cdot \sin\theta$$

$$= \frac{Eu^2 s}{(1+s^2)^{\frac{3}{2}}} + e - \frac{m}{2} \cdot 2c^2 \cdot u^4 \cdot E \cdot \sin\omega \cdot \cos\omega \cdot \sin\theta.$$

71. The value of $\frac{S-Ps}{h^2u^3}$ in the equation (*l*) of Art. 41, *Physical Astronomy*, is increased by

$$\overline{e-\frac{m}{2}}\,.\,2\,c^2\,.\,\frac{u}{h^2}\,E\,.\,\sin\omega\,.\,\cos\omega\,.\,\sin\theta.$$

$$\text{Now } \frac{E}{h^2}=\frac{E}{\mu}\,.\,\frac{\mu}{h^2}=\frac{E}{\mu}\,a:$$

and $u=a$, nearly:

$$\text{hence, } 2\,c^2\,.\,\frac{u}{h^2}\,E=2\,c^2\,.\,a^2\,.\,\frac{E}{\mu}, \text{ nearly.}$$

and the increase of $\frac{S-Ps}{h^2u^3}$ is nearly

$$2\,.\,\overline{e-\frac{m}{2}}\,.\,c^2a^2\,\frac{E}{\mu}\,\sin\omega\,.\,\cos\omega\,.\,\sin\theta.$$

Let the term which s receives in consequence be $A\sin\theta$. Upon substituting this in the equation, since

$$\frac{d^2(A\sin\theta)}{ds^2}+A\sin\theta=0,$$

the only term, it will be found, which receives an increment of the same form, is

$$\frac{m'u'^3s}{h^2u^4}\left(\frac{3}{2}+\frac{3}{2}\cos 2\,.\,\overline{\theta'-\theta}\right), \text{ or } n^2s\left(\frac{3}{2}+\frac{3}{2}\cos 2\overline{\theta'-\theta}\right),$$

nearly, the first part of which will be increased by $\frac{3n^2}{2}A\sin\theta$, n being here the same as m in the treatise on *Physical Astronomy*. Making then all the additional terms $=0$ which depend on $\sin\theta$, we have

$$\frac{3n^2}{2}A+2.\overline{e-\frac{m}{2}}\,.\,c^2a^2\,\frac{E}{\mu}\sin\omega\,.\,\cos\omega=0,$$

$$\text{or } A = \frac{-4}{3n^2}.\overline{e - \frac{m}{2}}.c^2 a^2.\frac{E}{\mu}\sin\omega.\cos\omega:$$

and the term in s, or in the tangent of the Moon's latitude,

$$= \frac{-4}{3n^2}.\overline{e - \frac{m}{2}}.c^2 a^2.\frac{E}{\mu}\sin\omega.\cos\omega.\sin\theta.$$

This is expressed in parts of the radius; the number of seconds will be found by multiplying it by $\frac{180 \times 60 \times 60}{\pi}$. And $ca = \frac{c}{\rho}$ nearly $=$ Moon's mean horizontal parallax. The term is, therefore,

$$\frac{4.60)''}{n^2.\pi}.\overline{e - \frac{m}{2}}.\sin^2\text{paral.}\frac{E}{\mu}\sin\omega.\cos\omega\sin\theta.$$

72. In investigating the alteration produced in u, it must be observed that the term

$$-\overline{e - \frac{m}{2}}.2c^2.\frac{u}{h^2}E.\sin\omega.\cos\omega.\cos\theta.s,$$

which is added to $\frac{T}{h^2u^3}$, upon putting for s its value $k\sin\overline{g\theta-\gamma}$, will contain the terms $\sin\overline{(g-1)\theta-\gamma}$, $\sin\overline{(g+1)\theta-\gamma}$, of which the former will be much increased by integration. And the term $-a\left(1-\frac{3s^2}{2}\right)$ in $-\frac{P}{h^2u^2}$, will contain $3as\times$ the term added to s, or

$$-\frac{4}{n^2}.k\sin\overline{g\theta-\gamma}.\overline{e-\frac{m}{2}}.c^2a^3.\frac{E}{\mu}.\sin\theta,$$

which will also produce terms of the same form. To find the alteration produced in $\frac{dt}{d\theta}$, it is necessary to include many

terms depending on $s^3 \cos\theta$, &c., and some terms among those added to P.

73. The disturbances in the Moon's motion produced by the Earth's ellipticity, though sensible, are very small. The ellipticity of Jupiter produces considerable disturbances in the motions of his satellites, and affects very much the progression of their apsides, and the regression of their nodes.

ON THE

METHODS OF ASCERTAINING

THE FIGURE OF THE EARTH

BY OBSERVATION.

74. The simplest of all methods, and the least dependent on theory, is that of measuring, by Geodetic operations, the distance between two places which are nearly in the same meridian, and observing the latitude of each. If two measurements of this sort be made in places whose difference of latitude is considerable, the ellipticity of the Earth may be ascertained; as we proceed to shew.

75. Prop. 32. The Earth being supposed a spheroid, to express the length of a small arc of the meridian at any point, in terms of the difference of latitude of its extremities.

Let $APQB$, fig. 19, be the meridian passing through the extremities P and Q of a small arc of latitude: at P and Q, draw normals meeting in R; then, R is very nearly the centre of curvature of p, which bisects PQ. And, if D be the difference of latitude of P and Q, $PQR = D$;

$$\therefore \; PQ = D \times pR.$$

$$\text{Now } pR = \frac{CK^2}{pF} = \frac{AC^2 \cdot BC^2}{pF^3} = \frac{a^2 c^2}{pF^3}.$$

putting a and c for the semi-axes.

$$\text{But } pF = \frac{c^2}{pL}; \quad \therefore \; pR = \frac{a^2 . pL^3}{c^4}.$$

And pLN = latitude of $p = l$; $\therefore$ $LN = pL \cos L$;

$$\therefore \; CN = \frac{a^2}{c^2} pL \cos l; \quad \text{and } Np = pL \sin l.$$

Substituting these in the equation $\dfrac{CN^2}{a^2} + \dfrac{Np^2}{c^2} = 1$, we get

$$\overline{pL}|^2 . \frac{a^2 \cos^2 l + c^2 \sin^2 l}{c^4} = 1;$$

$$\therefore pL^3 = \frac{c^6}{(a^2 \cos^2 l + c^2 \sin^2 l)^{\frac{3}{2}}},$$

$$\text{and } pR = \frac{a^2 c^2}{(a^2 \cos^2 l + c^2 \sin^2 l)^{\frac{3}{2}}}.$$

$$\text{hence } PQ = D . \frac{a^2 c^2}{(a^2 \cos^2 l + c^2 \sin^2 l)^{\frac{3}{2}}}.$$

If the ellipticity be small, let $a = c(1+e)$;

$$\therefore \; PQ = \frac{D . c^4 (1+2e)}{\{c^2(1+2e)\cos^2 l + c^2 \sin^2 l\}^{\frac{3}{2}}} = Dc(1 + 2e - 3e\cos^2 l),$$

$$\text{or } = D . c(1 - e + 3e \sin^2 l), \text{ nearly.}$$

76. Suppose then two arcs PQ, $P'Q'$, have been measured: suppose the latitudes of the middle to be l, l': the difference of the latitudes of P and Q to be D, that of P' and Q' to be D'. Then

$$\frac{PQ}{D} = c\,(1 - e + 3\,e\sin^2 l),$$

$$\frac{P'Q'}{D'} = c\,(1 - e + 3\,e\sin^2 l').$$

Subtracting, $\dfrac{P'Q'}{D'} - \dfrac{PQ}{D} = 3\,c\,e\,(\sin^2 l' - \sin^2 l)$;

$$\therefore\ e = \frac{\dfrac{P'Q'}{D'} - \dfrac{PQ}{D}}{3\,c\,(\sin^2 l' - \sin^2 l)}.$$

As e is small, we may, without sensible error, put for c, $\dfrac{PQ}{D}$, $\dfrac{P'Q'}{D'}$.

$$\text{hence } e = \frac{1 - \dfrac{PQ}{P'Q'} \cdot \dfrac{D'}{D}}{3\,(\sin^2 l' - \sin^2 l)}.$$

77. *Example.* By Lambton's measures in India, the arc of the meridian from lat. $8^0.9'.38'',4$ to lat. $10^0.59'.48''.9 =$ 1029100,5 feet.

By Svanberg's measures in Sweden, the arc of the meridian from lat. $65^0.31'.32''.2$ to lat. $67^0.8'.49''.8 =$ 593277,5 feet.

Here $PQ = 1029100{,}5$; $P'Q' = 593277{,}5$; $D = 10210'',5$; $D' = 5836'',6$; $l = 9^0.34'.44''$; $l' = 66^0.20'.10''$. Make

$$\frac{PQ}{P'Q'} \cdot \frac{D'}{D} = \cos^2\theta,$$

$$\text{or } 2\log\cos\theta = 20 + \log PQ + \log D' - \log P'Q' - \log D$$

$$= \log PQ + \log D' + \text{ar. com.} \log P'Q' + \text{ar. com.} \log D:$$

then, the numerator $= \sin^2\theta$. And the denominator

$$= 3\sin\overline{l' + l}\,.\,\sin\overline{l' - l}.$$

Hence $\log e = 2 \log \sin \theta - \log 3 - \log \sin \overline{l' + l} - \log \sin \overline{l' - l}$.

By calculating from the data above, $e = \dfrac{1}{306,3}$.

78. Attempts have also been made to determine the ellipticity of the Earth by measuring the distance between two places on the same parallel, and determining the difference of longitude, either by observations on Jupiter's satellites, or by observing the flash of gunpowder fired on a conspicuous place between them. The difference of longitude may also be determined by mere observation of angles, (see *Phil. Trans.* 1790).

79. Prop. 33. To express the distance of two places on the same parallel, in terms of their difference of longitude.

Let p, q, fig. 19, be the places; L their difference of longitude. Then, pq (which, when the arc is small, may be measured as a great circle without sensible error,) $= L \times CN$.

$$\text{Now } CN = \frac{a^2}{c^2} p L \cos l = \frac{a^2 . \cos l}{\sqrt{(a^2 \cos^2 l + c^2 \sin^2 l)}};$$

$$\therefore pq = L . \frac{a^2 \cos l}{\sqrt{(a^2 \cos^2 l + c^2 \sin^2 l)}} = L . c \cos l \, (1 + 2e - e \cos^2 l)$$

$$= L . c \cos l \, (1 + e + e \sin^2 l),$$

if the ellipticity be small.

80. If then one arc has been measured in the meridian, and another on a parallel, and if l be the latitude of the middle of the meridional arc, l' that of the parallel, we shall have these equations :

$$\frac{PQ}{D} = c\,(1 - e + 3e . \sin^2 l),$$

$$\frac{pq}{L} = c \cos l' \, (1 + e + e \sin^2 l').$$

Eliminating c, e may be found. This method is not considered to be practically accurate.

81. The method which on account of its great facility is now very extensively used, is that of observing the intensity of gravity in different latitudes, by means of the pendulum. It is usual to observe the number of vibrations made in a day by the same pendulum, in the different places at which it is proposed to compare the force of gravity; and likewise the number of vibrations made at London or Paris. The observations are commonly made in the manner described in Whewell's *Dynamics*, p. 255. Note: but in some experiments the clock-pendulum itself has been observed. The comparative number of vibrations being found, the comparative force of gravity, or the comparative length of the seconds' pendulum, can be found (Whewell's *Mechanics*, Art. 250: Wood's, Art. 300.): and, as the length of the seconds' pendulum has been very accurately determined at London and Paris, its length is known at all the places of observation. The French astronomers have used a method more direct, but less convenient, and probably less accurate: it is described at length in the Additions to Biot's *Astronomie Physique*, p. 138.

82. Let p and p' be the lengths of the seconds' pendulum in latitudes l and l', P that at the equator. Since these lengths are proportional to the intensities of gravity, we have, by (63),

$$\left.\begin{aligned} p &= P(1 + n\sin^2 l) \\ p' &= P(1 + n\sin^2 l') \end{aligned}\right\}, \quad \text{where } n = \frac{5m}{2} - e.$$

From these equations,

$$n = \frac{1 - \dfrac{p}{p'}}{\dfrac{p}{p'}\sin^2 l' - \sin^2 l} = \frac{1 - \dfrac{p}{p'}}{\sin^2 l' - \sin^2 l},$$

which may be calculated as the last example: then $e = \dfrac{5m}{2} - n$.

83. *Example.* At Madras,

$$l = 13^0 . 4' . 9'', \quad p = 39{,}0234.$$

At Melville Island,

$$l' = 74^0 . 47' . 12'', \quad p' = 39{,}2070.$$

$$\text{Hence, } n = {,}0053214, \text{ and } \frac{5\,m}{2} = {,}0086505\,;$$

$$\therefore\ e = {,}0033291 = \frac{1}{300}.$$

84. The ellipticity of the Earth has also been determined from the motion of the Moon. It appears from (71), that in the expression for the tangent of the Moon's latitude, there is this term

$$-e - \frac{m}{2} \cdot \frac{4 \cdot 60^3}{n^2 \cdot \pi} \cdot \sin^2 \text{parallax} \cdot \frac{E}{\mu} \cdot \sin \text{obliquity} \cdot \cos \text{obliquity} \cdot \sin\theta.$$

$$\text{Now } \frac{E}{\mu} = \frac{\text{Earth's mass}}{\text{Earth} + \text{Moon}} = \frac{70}{71}, \text{ nearly}: \ n = \frac{27{,}25}{365{,}25}:$$

mean horizontal parallax $= 57'$: obliquity $= 23^0 . 28'$, nearly: hence this term

$$= -e - \frac{m}{2} \cdot 4891'' \cdot \sin\theta.$$

It is found by observation that the coefficient $= -8''$;

$$\text{hence } e - \frac{m}{2} = \frac{8}{4891} = {,}001635. \quad \text{And } \frac{m}{2} = {,}001730:$$

$$\text{hence } e = {,}003365 = \frac{1}{297}.$$

The ellipticity, found by comparing the observed inequality in longitude with the calculated inequality, differs little from this.

85. The two latter methods, it will be observed, depend entirely upon the theory which we have laid down: the first and second are quite independent of theory. Their near agreement is one of the most convincing proofs that the principle of gravitation, and the suppositions upon which our theory is founded, are true.

86. For the calculation of parallax, it is necessary to know the distance of any point on the Earth's surface from the Earth's centre, and the angle ACp, (fig. 19.) which is called the *corrected* latitude: ALp being the *true* latitude, which = the elevation of the pole. The difference between the true and corrected latitude is called the angle of the center.

87. Prop. 34. To find the distance of any point on the Earth's surface from its center, in terms of the latitude of that point.

$$Cp^2 = CN^2 + Np^2 = \frac{a^4}{c^4} LN^2 + Np^2 = Lp^2 \cdot \left(\frac{a^4}{c^4} \cos^2 l + \sin^2 l\right) =$$

(putting for Lp^2 the value found in Prop. 32.)

$$\frac{a^4 \cos^2 l + c^4 \sin^2 l}{a^2 \cos^2 l + c^2 \sin^2 l}; \text{ and } Cp = \sqrt{\frac{a^4 \cos^2 l + c^4 \sin^2 l}{a^2 \cos^2 l + c^2 \sin^2 l}}.$$

If the ellipticity be small,

$$Cp = c\sqrt{\frac{(1+4e)\cos^2 l + \sin^2 l}{(1+2e)\cos^2 l + \sin^2 l}}$$

$$= c\,(1 + e\cos^2 l) \text{ nearly} = c\,(1 + e - e\sin^2 l).$$

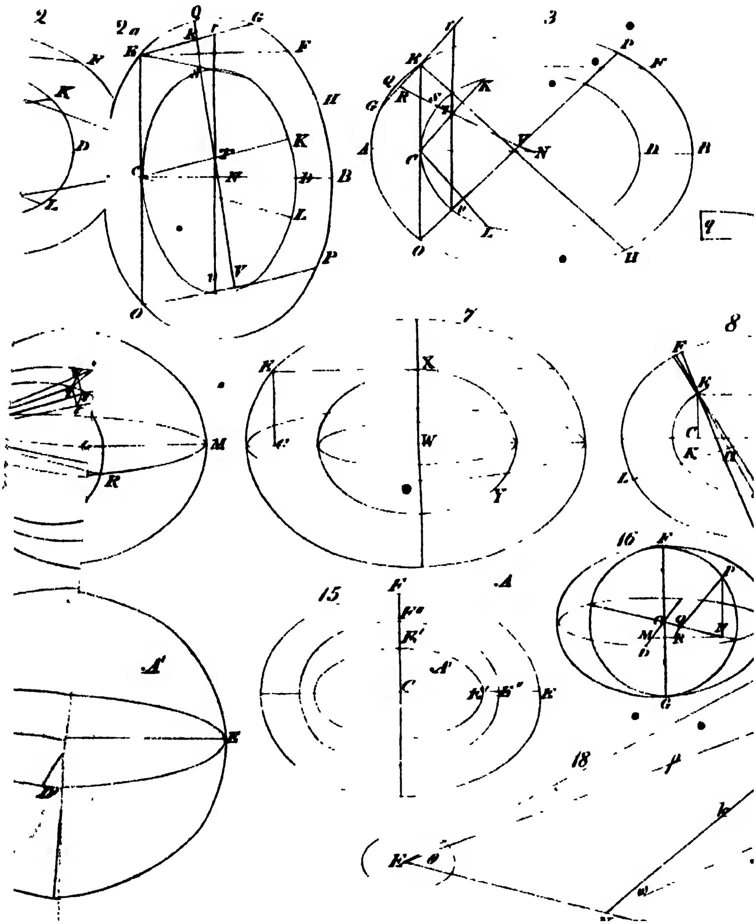
88. Prop. 35. To find the angle of the center.

Let ACp be the corrected latitude $= l'$.

$$\text{Then } \tan l = \frac{pN}{NL}; \quad \tan l' = \frac{pN}{CN};$$

$$\frac{\tan l'}{\tan l} = \frac{LN}{CN} = \frac{c^2}{a^2}, \text{ or } \tan l' = \frac{c^2}{a^2} \tan l.$$

$$\text{Hence,}\quad \tan(l-l') = \frac{\tan l - \tan . l'}{1 + \tan l . \tan l'}$$



85. The two latter methods, it will be observed, depend entirely upon the theory which we have laid down: the first and second are quite independent of theory. Their near

Hence, $\tan(l-l') = \dfrac{\tan l - \tan . l'}{1 + \tan l . \tan l'}$

$$= \frac{\left(1 - \frac{c^2}{a^2}\right)\tan l}{1 + \frac{c^2}{a^2}\tan^2 l} = \frac{(a^2 - c^2)\sin l \cos l}{a^2 \cos^2 l + c^2 \sin^2 l}.$$

When the ellipticity is small, putting $l - l'$ for $\tan(l - l')$,

$$l - l' = 2e . \sin l . \cos l \text{ nearly} = e . \sin 2l.$$

This is in parts of the radius: the number of seconds is

$$\frac{180 . 60 . 60}{\pi} e . \sin 2l.$$

PRECESSION OF THE EQUINOXES,

AND

NUTATION OF THE EARTH's AXIS.

ON THE COMPOSITION OF ROTATORY MOTION.

1. Prop. 1. If a body revolve about an axis AB, (fig. 1.) with an angular velocity ω, and if a force be impressed upon it which would make it revolve about the axis AC, with an angular velocity ω'; then the body will not revolve about either of the axes AB, AC, but about an axis AD, in the plane BAC, dividing the angle BAC so, that $\sin BAD : \sin CAD :: \omega' : \omega$.

2. It is evident, that the new axis of rotation is that line in the body, which, when the effect of both the original motions is considered, remains at rest. If, then, a line AD in the plane BAC be that axis, the angular motion about AB, would tend to raise any point in it, as D, *above* the plane of the paper, as much as the angular motion about AC, would tend to depress it *below* the plane of the paper. From D, draw DE, DF, perpendicular to AB, AC. In consequence of the angular motion about AB, the point D would be raised above the paper, with the velocity

$$\omega \times ED = \omega \times AD \cdot \sin BAD.$$

And, in consequence of the angular motion about AC, D would be depressed below the paper, with the velocity

$$\omega' \times FD = \omega' \times AD \cdot \sin CAD.$$

Making these equal,

$$\omega \times AD . \sin BAD = \omega' \times AD . \sin CAD;$$

$$\therefore \sin BAD : \sin CAD :: \omega' : \omega.$$

3. Now, to shew that AD really is the axis of rotation, take any point P in the body: with centre A, suppose a spherical surface described, passing through P, and cutting AB, AC, AD, in B, C, D; let BDC, BP, CP, be arcs of great circles. Let PQ, drawn perpendicular to PB on the surface of the sphere, be the motion of P, produced by the rotation about AB only, in the very small time t''; let PR, drawn perpendicular to PC, on the surface of the sphere, be the motion of P, produced by the rotation about AC only, in the same time; then, if the parallelogram QR be completed, PS, the diagonal, is the true motion of P, in that time. Now $CPR = 90^0 = BPQ$; adding RPB to both, $CPB = RPQ$. Also

$$\sin SPQ : \sin SPR :: \sin SPQ : \sin PSQ :: SQ : PQ$$

$$:: PR : PQ.$$

$$\text{But } PR = \omega' t \times \text{Sin } PC \text{ to radius } AC;$$

$$PQ = \omega t \times \text{Sin } PB;$$

$$\therefore \sin SPQ : \sin SPR :: \omega' . \text{Sin } PC : \omega . \text{Sin } PB.$$

$$\text{And } \sin BPD : \sin CPD :: \frac{\text{Sin } BD . \sin BDP}{\text{Sin } BP} :$$

$$\frac{\text{Sin } CD . \sin CDP}{\text{Sin } CP} :: \text{Sin } PC . \text{Sin } BD : \text{Sin } PB . \text{Sin } CD$$

$$:: \omega' . \text{Sin } PC : \omega . \text{Sin } PB,$$

(since Sin BD : Sin CD :: sin BAD : sin CAD :: ω' : ω).

Hence, sin SPQ : sin SPR :: sin BPD : sin CPD.

Since, then, the two angles CPB, RPQ, are equal, and are divided into parts, whose sines are in the same ratio, it follows,

that those parts are equal, or $BPD = QPS$. Adding to each BPS,

$$DPS = BPQ = 90^{\circ};$$

and, therefore, PS is perpendicular to the plane ADP. In the same manner it may be shewn, that the plane passing through any other point of the body, and through AD, is perpendicular to the motion of that point; and, since the axis of rotation is the line of intersection of all the planes perpendicular to the motion of every point, AD must be the axis of rotation.

4. It is here supposed, that the angular motion about AB, tends to raise all the particles between AD and AC, and that the angular motion about AC tends to depress them. If, however, the angular motions about both AB and AC, (fig. 2), tend to raise the particles between AB and AC, produce CA to C': then the angular motion about AC' tends to raise the particles between AC' and AB, and the angular motion about AB tends to depress them. Hence, the new axis of rotation will be the line AD, which makes $\sin BAD : \sin C'AD :: \omega' : \omega$. The same is true, if both angular motions tend to depress the particles between AB and AC.

5. Prop. 2. The angular velocity about the new axis AD, will be to the original angular velocity about AB, as $\sin BAC$ to $\sin DAC$; and the angular velocity about AD to the original angular velocity about AC, as $\sin BAC$ to $\sin BAD$.

6. Let ω'' be the angular velocity about AD;

$$\text{then, } PS = \omega'' t \,.\, \text{Sin } DP.$$

$$\text{Now } PQ : PS :: \sin PSQ : \sin PQS :: \sin SPR : \sin QPR$$

$$:: \sin DPC : \sin BPC,$$

since BP, DP, CP, are perpendicular to PQ, PS, PR, respectively. But

$$\sin DPC : \sin BPC :: \frac{\text{Sin } DC \,.\, \sin DPC}{\text{Sin } DP} : \frac{\text{Sin } BC \,.\, \sin BCP}{\text{Sin } BP};$$

putting then for PQ and PS their values,

$$\omega t \cdot \text{Sin } BP : \omega'' t \cdot \text{Sin } DP :: \frac{\text{Sin} DC}{\text{Sin} DP} : \frac{\text{Sin } BC}{\text{Sin } BP};$$

whence, $\omega : \omega'' :: \text{Sin } DC : \text{Sin } BC :: \sin DAC : \sin BAC.$

And, since $\omega' : \omega :: \sin BAD : \sin DAC$ by (2);

$$\therefore \omega' : \omega'' :: \sin BAD : \sin BAC.$$

7. We have supposed, in the enunciations of the Propositions above, that an angular motion about one axis, is suddenly impressed upon a body which had previously an angular motion about another axis. It is evident, that the conclusions are the same, if we suppose both angular motions to be impressed at once.

8. From these Propositions, compared with Prop. 4, Wood's *Mechanics*, or Articles 26 and 27, Whewell's *Mechanics*, it appears, that if two forces in the directions AB, AC, be proportional to ω, ω', their resultant will be in the direction AD, and will be proportional to ω''. And hence, if several angular motions were impressed upon a body at the same time, the new axis of rotation and the angular velocity about that axis would be found, by finding the direction and magnitude of the resultant of forces in the directions of the several axes of rotation, and proportional to the angular velocities.

9. If then a body revolve about an axis, and angular motions about two other axes be impressed upon it, it is indifferent whether we first compound the two impressed motions, and then compound their resultant with the original motion, or compound the original motion with one of the impressed motions, and then compound their resultant with the other motion. For, in compounding the forces proportional to these angular velocities, the order in which they are taken is indifferent.

10. If the angular motion about AC be not produced instantaneously, but by the continued action of a finite force, its

effect may be found, by supposing the time divided into a great number of small intervals, and supposing the angular velocity generated in each of those intervals to be impressed at the end of each, and then finding the limit to which we approach, by increasing indefinitely the number of these intervals.

11. Prop. 3. If a uniform force act upon the body, tending to give it a motion of rotation about an axis which is always perpendicular to the axis about which it is at each instant revolving, and always in the plane BAC, (fig. 1 and 3), the angular velocity will be unaltered.

Let ω be the original angular velocity; and suppose the impressed force such as would generate in 1″, the angular velocity a. Let this 1″ be divided into n parts; then the angular velocity generated in each of these parts, is $\frac{a}{n}$. Compounding the angular velocities ω and $\frac{a}{n}$, of which the axes AB, AC, (fig. 3.) are at right angles to each other, we find, by (7), the new angular velocity about $AD = \sqrt{\omega^2 + \frac{a^2}{n^2}}$. Compounding this angular velocity with the angular velocity $\frac{a}{n}$, generated in the second small interval, and observing that the axis Ac, about which it is produced, is perpendicular to the axis AD, about which the body is now revolving, the angular velocity at the end of the second interval, is

$$\sqrt{\omega^2 + \frac{a^2}{n^2} + \frac{a^2}{n^2}} = \sqrt{\omega^2 + \frac{2a^2}{n^2}}.$$

In the same way the angular velocity at the end of the third interval, is

$$\sqrt{\omega^2 + \frac{2a^2}{n^2} + \frac{a^2}{n^2}} = \sqrt{\omega^2 + \frac{3a^2}{n^2}}, \text{ \&c.};$$

that at the end of the n^{th} interval, or at the end of 1″, is

$$\sqrt{\omega^2 + \frac{n a^2}{n^2}} = \sqrt{\omega^2 + \frac{a^2}{n}}.$$

Let n be increased without limit, and this becomes $= \sqrt{\omega^2} = \omega$. The angular velocity, therefore, is not altered in the first 1″; and since the same demonstration applies to every succeeding 1″, it follows, that the angular motion is unaltered.

12. PROP. 4. Under the same circumstances, the axis of rotation has a uniform motion in space, from the position AB towards AC; and the angle described in 1″ $= \frac{a}{\omega}$.

Suppose 1″ divided into n parts, as in the first Proposition, and suppose AB, AC, (fig. 1), to be at right angles. If we suppose the angular velocity $\frac{a}{n}$, about the axis AC, to be impressed instantaneously, and suppose AD to be the new axis of revolution; then, by Prop. 1,

$$\frac{\sin BAD}{\sin CAD} = \frac{a}{n\omega},$$

or, in the present case,

$$\frac{\sin BAD}{\cos BAD} = \frac{a}{n\omega}, \text{ or } \tan BAD = \frac{a}{n\omega}.$$

Suppose n very much increased; then $\tan BAD$ being diminished without limit, we may put the arc for the tangent; hence, $BAD = \frac{a}{n\omega}$. And since, by the last Proposition, the angular velocity remains unaltered, angles equal to BAD will be added to BAD in each succeeding interval; and, therefore, at the end of 1″, the axis of revolution will be inclined to the line, which was the axis of revolution at the beginning of that 1″, by the

angle $\frac{a}{\omega}$. Since the same is true of every successive 1″, the axis of revolution will move from the position AB, towards the position AC, with the angular velocity $\frac{a}{\omega}$.

13. Prop. 5. Under the same circumstances, if a spherical surface be described in the body about the point A, at which the axes intersect each other, the points at which the successive axes of revolution cut this surface, will lie in the circumference of a small circle, whose radius = radius of the sphere $\times \frac{a}{\omega^2}$, nearly.

Let AB, (fig. 3), be the original axis, and, as before, suppose 1″ divided into n parts, and the angular velocity $\frac{a}{n}$, about an axis perpendicular to the axis of rotation, to be impressed at the end of each. At the end of the first interval, the axis will be transported from AB to AD, the angle BAD being $= \frac{a}{n\omega}$; and at the end of the second and succeeding intervals, it will have the positions AD', AD'', &c. in space, each of the angles DAD', $D'AD''$, &c. being $= \frac{a}{n\omega}$. Now, in $\frac{1''}{n}$, the body revolves through $\frac{\omega}{n}$; hence, if on the surface of the sphere, the angle $D'Dd'$ be made $= \frac{\omega}{n}$, and $Dd' = DD'$, d' is the point which, at the end of the second interval, coincides with D'; and Ad' is, therefore, the line in the body, which, at the end of the second interval, is the axis of rotation. In the same manner, if $d'd''$ make with Dd' produced the angle $\frac{\omega}{n}$, and $d'd'' = D'D''$, Ad'' is the line, which, at the end of the third interval, is the axis of rotation, &c. From the construction it is evident, that BD, Dd', $d'd''$, &c., are the sides of a regular polygon. Suppose

now, n increased without limit, or the number of the sides of the polygon increased without limit; the limit of the line traced on the spherical surface by its intersections with the successive axes, is a circle.

14. To find the radius of this circle, we observe, that if the circle and polygon be small, the sum of all the angles at D, d', &c. $= 2\pi$; but, since each of them $= \frac{\omega}{n}$, their number, or the number of sides of the polygon $= \frac{2n\pi}{\omega}$. And the length of each $= AB \times \frac{a}{n\omega}$; therefore the circumference of the polygon, or ultimately of the circle $= \frac{2\pi . AB . a}{\omega^2}$; therefore the radius of the circle $= AB \times \frac{a}{\omega^2}$.

15. If the force which acts upon the body be nearly, but not exactly, uniform, and if the axes about which it tends to produce motion, be not contained in the plane BAC, then the propositions above will be nearly, but not exactly, true. The line traced on the spherical surface in Prop. 5, by the successive poles of rotation, will be a spiral approaching very nearly to a circle: the change of position of the axis in space at every instant, will be in the plane passing through the axis of rotation, and the axis of impressed motion at that instant; and the velocity of the change, though not uniform, will be at that

ON PRECESSION AND NUTATION.

16. Prop. 6. To explain the physical cause of solar precession, and solar nutation.

Let A, (fig. 4), be the Earth's centre; AB the axis of rotation; S the Sun; CHG the equator of the terrestrial spheroid; CAG that diameter of the equator, which is perpendicular to AS; and suppose the Earth to be in the position which it has at the summer solstice. In the succeeding investigations, which relate only to the motion of the Earth about its center of gravity, we may suppose the center of gravity to be kept at rest, and the motion of the Earth about this point will be the same, as if we supposed it moving freely, (Whewell's *Dynamics*, Art. 129, Poisson, *Mecanique*, 402). Suppose, then, A the Earth's center, to be at rest; and consider the effect which the Sun's attraction would then produce on the Earth. If the Earth were spherical, it is evident that the Sun's attraction would have no tendency to give the Earth any rotation about the center A. But the Earth is an oblate spheroid; we must, therefore, consider the effect produced by the Sun's attraction on the parts of the spheroid which are exterior to the sphere touching the spheroid at its poles. Now the Sun's attraction is inversely as the square of the distance of the matter attracted; and, consequently, the attraction on the spheroidal protuberance at K, is greater than the attraction on that at H. The effect of this, supposing the Earth at rest, would evidently be to give it a motion of rotation about the line CAG, in such a direction as to bring the point K towards B. But the Earth is not at rest, but is revolving about the axis AB, which nearly coincides with the axis of the spheroid, in such a direction as to carry the point C towards K. This, then, is exactly the case considered in Prop. 4. The Earth has a previous motion of rotation; and a force acts on it, which, for a short time at least, is uniform, and which tends to give it a motion of rotation round an axis

perpendicular to the axis about which it is revolving. The axis of rotation, therefore, moves from the position AB towards AC, describing in each $1''$ the angle $\frac{a}{\omega}$. Let AD be the position of the axis of rotation after a short time; and let AL be perpendicular to the ecliptic. It is evident, that the path of the pole, or the arc joining BD, is a tangent to the circle passing through B, whose center is in AL, and whose plane is perpendicular to AL; and that the motion of the pole in this circle, is in a direction opposite to the direction of the Earth's rotation, or is retrograde. This, then, is precession of the equinoxes.

17. If we now consider the situation of the Earth at the winter solstice, (fig. 5), it will be seen, that the Sun's attraction upon H, is now greater than the attraction on K; and, therefore, the motion of rotation about CAG, which the Sun's action tends to produce in the Earth, is in the same direction as before. The motion of the axis of rotation is, therefore, in the same direction as at the summer solstice.

18. If the effect of the Sun's action be examined in any other situation of the Earth, it will be found that, as the small line BD is always perpendicular to the plane passing through the Sun and the Earth's axis, it is not always a tangent to the small circle whose center is in AL. The Sun's action, therefore, sometimes increases the inclination of the axis of rotation to the axis of the ecliptic, and sometimes diminishes it; but, (as we shall show hereafter), does not permanently alter it. This phænomenon is one part of solar nutation. The angular motion of the axis of rotation about the axis of the ecliptic, is always in the same direction; but as the action of the Sun is different in different positions of the Earth, and is 0 at the equinoxes, this angular motion or precession is irregular. The correction which it is necessary to apply to a uniform precession, is the other part of solar nutation.

19. PROP. 6. *To explain the physical cause of lunar precession, and lunar nutation.*

Since the Moon describes, (very nearly), a great circle about the Earth in a month, in the same manner as the Sun in a year, the same explanation which has been given for the precession and nutation produced by the Sun in a year, will apply to those produced by the Moon in a month. But the monthly nutation produced by the Moon is so small, that it is very seldom considered. Since, however, the magnitude and direction of the permanent precession produced by the Sun, depends upon the inclination of the Earth's axis to the axis of the ecliptic, or Sun's apparent orbit, it is easy to see, that the magnitude and direction of the precession produced by the Moon in one month, depends upon the inclination of the Earth's axis to the axis of the Moon's orbit. Now this is perpetually varying; the axis of the Moon's orbit revolves about the axis of the ecliptic in about 18 years, 7 months, with a motion nearly uniform, and preserving nearly a constant inclination. The velocity and direction of the motion of the Earth's axis, produced by the Moon, is, therefore, irregular. We shall shew, that the precessional motion, though irregular, is permanent; but that the alteration in the inclination to the axis of the ecliptic is periodical; the inclination returning to its former value, in a revolution of the Moon's nodes. This change in the inclination, is one part of lunar nutation; the other part is the correction which must be applied to the mean precession, in order to find the true.

20. Prop. 7. The velocity of the Earth's rotation is unaltered by the action of the Sun and Moon.

Since the Sun's action would give the Earth a motion of rotation about an axis, in the plane of the equator of the terrestrial spheroid, and the Moon's action would give it a rotation about another axis in the same plane, their combined action would give the Earth a rotation about a third axis in that plane, by (7).

Now to shew that the Earth's angular velocity is unaltered, we must shew, that this axis is always perpendicular to the axis

of rotation. Let AC, (fig. 3), be this axis, AB the axis of rotation; by (15), the points of intersection of this axis with a sphere described in the Earth about A, lie nearly in a small circle, whose center is E. The ellipticity of the Earth is produced by its rotation; and since the axis of rotation passes successively through all points of the circle BF in one revolution, the axis of the spheroid will pass through E, the center of that circle. AE, therefore, is perpendicular to AC; or if EC be joined by an arc of a great circle, EC is a quadrant. And EBC is a right angle; hence, BC is also a quadrant, or BAC is always a right angle. Consequently, by Prop. 3, the velocity of rotation is not altered.

21. PROP. 8. To calculate the value of a; the force acting on the Earth being the attraction of a distant body; and the Earth being a homogeneous spheroid.

Let A, (fig. 6), be the Earth's center; AB the axis of the spheroid: S the attracting body; take P, the projection of any point of the Earth, and draw PN perpendicular to SA, and PM perpendicular to the projection of the equator. Let f be the attraction of S upon A; then the attraction of S upon P, is $f.\dfrac{SA^2}{SP^2}$, or, if $SR = \dfrac{SA^3}{SP^2}$, it $= f.\dfrac{SR}{SA}$. If SA, then, be taken to represent the force f, SR will represent the force on P, and RA the difference of forces on P and A; or that difference of forces $= f.\dfrac{RA}{SA}$, and is in the direction RA. Now if a force f were applied to every point of the spheroid, in the direction SA, it would produce no effect in giving the Earth a motion about A; without altering the motion, therefore, we may suppose this force applied; that is, we may suppose the only force acting at each point, to be the difference of the force really acting there, and the force at A. In the figure, the point A is evidently the projection of the axis, about which these forces would make the Earth revolve. We must, therefore, find the momentum of the force $f\dfrac{RA}{SA}$ impressed on a particle δm at P, about the centre A.

Let $SQ = SA$; then, S being very distant, AQ is nearly perpendicular to SA; and

$$SR = \frac{SQ^3}{(SQ - PQ)^2} = SQ + 2PQ, \text{ nearly;}$$

or $RQ = 2PQ$. The force $f \cdot \frac{RA}{SA} \delta m$ may now be resolved into

$$f \frac{RQ}{SA} \delta m \text{ and } f \cdot \frac{QA}{SA} \delta m, \text{ or } f \delta m \cdot \frac{2PQ}{SA} \text{ and } f \delta m \cdot \frac{PN}{SA},$$

acting at P in the directions QP, PN. Their momenta to turn the Earth in the direction KB, are

$$-f \delta m \cdot \frac{2PQ \cdot PN}{SA}, \text{ and } -f \delta m \cdot \frac{PN \cdot PQ}{SA},$$

(considering PQ parallel to NA); the sum of these

$$= -3f \cdot \delta m \cdot \frac{PN \cdot PQ}{SA}.$$

If the absolute force of the attracting body $S = S$, and $SA = r$, then $f = \frac{S}{r^2}$, and the moment of the force on P

$$= -\frac{3S}{r^3} \delta m \cdot PN \cdot PQ.$$

$$\text{Let } AM = x; \quad MP = y; \quad BAS = \theta.$$

$$\text{Then } PN = MV - MT = y \sin\theta - x \cos\theta;$$

$$PQ = PV + AT = y \cos\theta + x \sin\theta.$$

Substituting these values, the moment of the force on the particle at P

$$= \frac{3S}{r^3} \delta m \cdot \{(x^2 - y^2) \sin\theta \cos\theta + xy (\cos^2\theta - \sin^2\theta)\}.$$

22. We must now find the sums of the expressions $x^2\delta m$, $y^2\delta m$, $xy\delta m$, for every particle of the spheroid. Suppose the spheroid divided into slices by planes parallel to the plane of xy; let two of these be at the distances z, and $z+\delta z$ respectively; z being measured perpendicular to the plane of xy, and δz being small; and let the included slice be divided into prisms, by planes parallel to yz; two of these being at the distances x, and $x+\delta x$ from the plane of yz; and take a portion of this prism, included between the co-ordinates y and $y+\delta y$. The volume of this portion $=\delta x \,.\, \delta y \,.\, \delta z$; and if k be the density, the expression $xy\delta m$ becomes, for this portion, $\delta z \,.\, \delta x \,.\, kxy\delta y$. If p be the sum of $xy\delta m$ for the prism,

$$\frac{dp}{dy} = \delta z \,.\, \delta x \,.\, kxy,$$

$$\text{or } p = \delta z \,.\, \delta x \,.\, \frac{kxy^2}{2};$$

taking this between the limits

$$y = -\frac{c}{a}\sqrt{a^2 - x^2 - z^2}, \text{ and } y = +\frac{c}{a}\sqrt{a^2 - x^2 - z^2},$$

(since the equation to the surface of the spheroid is

$$\frac{x^2+z^2}{a^2} + \frac{y^2}{c^2} = 1), \quad p = 0.$$

Hence, the sum of $xy\delta m$ for the whole spheroid, is $=0$

23. For the sum of $x^2\delta m$ it appears in the same manner, that if w be the sum for the prism,

$$\frac{dw}{dy} = \delta z \,.\, \delta x \,.\, kx^2, \text{ or } w = \delta z \,.\, \delta x \,.\, kx^2 y;$$

which, taken between the limits $\mp \frac{c}{a}\sqrt{a^2 - x^2 - z^2}$, gives

$$w = 2k\delta z \,.\, \delta x \,.\, \frac{c}{a} x^2 \sqrt{a^2 - x^2 - z^2}.$$

This is ultimately the increment of the sum of $x^2 \delta m$ for the slice, produced by giving to x the increment δx; calling this sum v,

$$\frac{dv}{dx} = 2k\delta z \cdot \frac{c}{a} \cdot x^2 \sqrt{a^2 - z^2 - x^2},$$

$$\text{and } v = 2k\delta z \frac{c}{a} \int_x x^2 \sqrt{a^2 - z^2 - x^2},$$

z being considered constant in the integration.

The integral is

$$k\delta z \cdot \frac{c}{a} \cdot \left\{ -\frac{x}{2}(a^2 - z^2 - x^2)^{\frac{3}{2}} + \frac{a^2 - z^2}{4} x \sqrt{a^2 - z^2 - x^2} \right.$$
$$\left. + \frac{(a^2 - z^2)^2}{4} \sin^{-1} \cdot \frac{x}{\sqrt{a^2 - z^2}} \right\}.$$

The limits of x are the least and greatest values of x in the slice; that is, the values given by the equation to the surface, upon making $y = 0$; they are, therefore, $\mp \sqrt{a^2 - z^2}$; and

$$v = k\delta z \cdot \frac{c}{a} \cdot \frac{(a^2 - z^2)^2}{4} \cdot \pi.$$

Now, if u be the sum of $x^2 \delta m$ for the whole spheroid, v is ultimately the increment of u, arising from giving to z the increment δz; hence,

$$\frac{du}{dz} = k \cdot \frac{c}{a} \cdot \frac{\pi}{4} \cdot (a^2 - z^2)^2 = k \cdot \frac{c}{a} \cdot \frac{\pi}{4} \times (a^4 - 2a^2 z^2 + z^4)$$

$$\therefore u = k \cdot \frac{c}{a} \cdot \frac{\pi}{4} \cdot \left(a^4 z - \frac{2}{3} a^2 z^3 + \frac{z^5}{5} \right);$$

taking this between the limits $z = \mp a$,

$$u = k \cdot \frac{c}{a} \cdot \pi \cdot \frac{4a^5}{15} = \frac{4\pi}{15} k a^2 c \times a^2.$$

24. In the same manner, it would be found that the sum of the $y^2 \delta m$ for the spheroid $= \frac{4\pi}{15} ka^2 c \times c^2$. Hence, the moment of all the impressed forces

$$= \frac{3S}{r^3} \cdot \frac{4\pi}{15} ka^2 c (a^2 - c^2) \sin\theta \cdot \cos\theta.$$

25. Now, to find the angular velocity which this would generate in 1″ about the axis whose projection is A, we must (Whewell's *Dynamics*, Art. 74.) divide this by the moment of inertia, or the sum of $(x^2 + y^2)\delta m$ for the whole spheroid. This sum is found in the same manner to be

$$= \frac{4\pi}{15} ka^2 c (a^2 + c^2).$$

Hence, the angular velocity generated in 1″, or α,

$$= \frac{3S}{r^3} \cdot \frac{a^2 - c^2}{a^2 + c^2} \cdot \sin\theta \cdot \cos\theta.$$

26. Prop. 9. To calculate α, supposing the earth heterogeneous.

Suppose the Earth composed of strata of different densities, bounded by spheroidal surfaces of different ellipticities, as in the Treatise on the Figure of the Earth. Let c be the semi-axis of any one of these spheroids, e its ellipticity; c the semi-axis of the external surface, e its ellipticity; ρ the density at any point; ρ and e being functions of c. Then, (since $a^2 - c^2 = 2c^2 e$ nearly, and $a^2 + e^2 = 2c^2$ and $a^2 c = c^3$ nearly) as in Prop. 22. of the Figure of the Earth, we shall have for the moment of the impressed forces,

$$\frac{3S}{r^3} \sin\theta \cdot \cos\theta \cdot \frac{8\pi}{15} \cdot \int_c \rho \cdot \frac{d(c^5 e)}{dc}$$

$$= \frac{3S}{r^3} \sin\theta \cdot \cos\theta \cdot \frac{8\pi}{15} \psi(\text{c}).$$

And the moment of inertia

$$=\frac{8\pi}{15}\int_{c}\rho\,.\,\frac{d(c^5)}{dc}=\frac{8\pi}{15}\,\sigma(c),\ \text{if}\ \sigma(c)=\int_{c}\rho\,.\,\frac{d(c^5)}{dc}.$$

Hence, the Earth being heterogeneous,

$$a=\frac{3S}{r^3}\,.\sin\theta\,.\cos\theta\,.\frac{\psi(c)}{\sigma(c)}.$$

27. In the investigation of the value of a, we have supposed that the only force which tends to give the Earth a rotatory motion about AC, fig. 3 and 4, is the action of a distant body. This, however, is not strictly true; for, since AD, the axis about which the Earth is at any instant revolving, does not coincide with AE the axis of the figure, the centrifugal force will diminish the effect produced by the distant body. With an ellipticity, however, so small as that of the Earth, this diminution is not sensible.

28. Prop. 10. To investigate the quantity of solar precession for any given time.

Suppose EC, fig. 7, to be the projection of the ecliptic on the surface of a sphere described about the Earth's center; S, P, Q, the projections of the Sun's place, the pole of the Earth, and the pole of the ecliptic; join SP by an arc of a great circle. By Prop. 8 and 9, the value of a is

$$\frac{3B\,.\,S}{r^3}\sin\theta\,.\cos\theta,$$

B being $=\dfrac{a^2-c^2}{a^2+c^2}$ if the Earth be homogeneous, and $=\dfrac{\psi(c)}{\sigma(c)}$ if the Earth be heterogeneous. Now, θ is the angle made by the Earth's axis with the line joining the centers of the Sun and Earth, and is, therefore, in fig. 7, represented by SP: hence

$$a=\frac{3A\,.\,S}{r^3}\sin SP\,.\cos SP:$$

and, by Prop. 4, the pole of rotation moves with the velocity

$$\frac{a}{\omega} \text{ or } \frac{3B.S}{r^3\omega} \sin SP.\cos SP,$$

in the direction Pp perpendicular to PS. The resolved part of this motion, perpendicular to PQ, or parallel to CE, is

$$\frac{3B.S}{r^3\omega}.\sin SP.\cos SP.\cos SPC.$$

Let ES, the Sun's longitude $= l$; QP the inclination of the equator and ecliptic $= I$. Then

$$\cos SP = \cos CS.\cos CP = \sin I.\sin l;$$

$$\sin SP = \frac{\sin PC}{\sin PSC} = \frac{\cos I}{\sin PSC};$$

$$\cos SPC = \cos SC.\sin PSC = \sin l.\sin PSC;$$

therefore, the velocity of the pole parallel to CS is

$$\frac{3B.S}{r^3\omega} \sin I.\cos I.\sin^2 l,$$

and the motion of the pole in that direction

$$= \frac{3B.S}{\omega} \sin I.\cos I.\int_t \frac{\sin^2 l}{r^3} = \frac{3B.S}{\omega} \sin I.\cos I.\int_l.\frac{\sin^2 l}{r^3}.\frac{dt}{dl}$$

Now, by Art. 12 of *Physical Astronomy*, neglecting the Earth's mass in comparison with the Sun's,

$$\frac{dt}{dl} = \frac{r^2}{\sqrt{a(1-e^2).S}},$$

where a and e are the semi-axis-major and excentricity of the Sun's apparent orbit; therefore, the motion of the pole parallel to CS

$$= \frac{3\,B\sqrt{S}}{\omega\sqrt{a\,(1-e^2)}}\,\sin I\,.\,\cos I\,.\int_t.\,\frac{\sin^2 l}{r}$$

$$= \frac{3\,B\sqrt{S}}{\omega\,(a\,.\,\overline{1\,-\,e^2})^{\frac{3}{2}}}\,\sin I\,.\,\cos I \int_t \sin^2 l\,(1+e\cos\overline{l-k})$$

where k is the longitude of the Sun's perigee. Let $T=1$ year: by Art. 15, of *Physical Astronomy,*

$$T=\frac{2\,\pi\,.\,a^{\frac{3}{2}}}{\sqrt{S}}:$$

our expression becomes then

$$\frac{6\,\pi\,.\,B}{T\omega\,.\,(1-e^2)^{\frac{3}{2}}}\,\sin I\,.\,\cos I \int_t \sin^2 l\,(1+e\cos\overline{l-k})$$

$$= \frac{3\,\pi\,.\,B}{T\omega\,(1-e^2)^{\frac{3}{2}}}\,\sin I\,.\,\cos I$$

$$\times\left(C+l-\frac{\sin 2\,l}{2}+e\,.\sin\overline{l-k}-\frac{e}{2}\sin\overline{l+k}-\frac{e}{6}\sin\overline{3\,l-k}\right).$$

The three last terms of this expression are so small when numerically calculated, that they are rejected; and the motion of the pole parallel to CS, is, therefore,

$$\frac{3\,\pi\,.\,B}{T\omega\,(1-e^2)^{\frac{3}{2}}}\,.\,\sin I\,.\cos I\left(C+l-\frac{\sin 2\,l}{2}\right),$$

$$\text{or }\ \frac{3\,\pi\,.\,B}{T\omega}\,\sin I\,.\cos I\left(C+l-\frac{\sin 2\,l}{2}\right)\ \text{nearly.}$$

29. The first term of this expression never changes sign, but increases as l increases, and is, therefore, nearly proportional to the time from any fixed epoch; it is the precessional motion of the pole. The second term is periodical, going through all its values in half a year: it is the correction which must be applied to the place of the pole found on the sup-

position of uniform precession, in order to obtain its true place. This is one part of Solar Nutation.

30. To obtain the precession of the equinoxes, or the angular motion of P about Q, we must divide the expression above by $\sin I$: it is, therefore,

$$\frac{3\pi . B}{T\omega} \cos I . \left(C + l - \frac{\sin 2l}{2}\right).$$

The first term is the uniform precession; the second is the correction to be applied to it, called the Solar Equation of the Equinoxes in longitude.

31. Prop. 11. To investigate the change in the inclination of the ecliptic, produced by the Sun's action on the Earth.

In (28) it was found that the velocity of the pole in the direction Pp, fig. 7, is

$$\frac{3B.S}{r^3\omega} \sin SP . \cos SP;$$

therefore, its velocity in the direction PQ

$$= \frac{3B.S}{r^3\omega} \sin SP . \cos SP . \sin SPC.$$

But $\sin SP . \sin SPC = \sin SC = \cos l$;

$\cos SP = \cos PC . \cos SC = \sin I . \sin l$;

therefore, the motion in the direction PQ, or the diminution of the inclination,

$$= \frac{3B.S}{\omega} \sin I \int_t \frac{\sin l . \cos l}{r^3};$$

which, as in (28), is changed into

$$\frac{6\pi . B}{T\omega} \sin I \int_l \sin l \cos l \,(1 + e \cos \overline{l - k}).$$

Neglecting, as before, the terms depending on e, this is

$$-\frac{3\pi . B}{T\omega}\sin I\frac{\cos 2l}{2}:$$

or, if I be the mean inclination, the true inclination

$$=I+\frac{3\pi . B}{T\omega}\sin I . \frac{\cos 2l}{2}.$$

The term added to I is the second part of Solar Nutation.

32. If we call the two parts of Solar Nutation, mentioned in (28) and (31), x and y, we shall easily perceive that they are connected by this equation,

$$x^2\left(\frac{2T\omega}{3\pi . B . \sin I . \cos I}\right)^2+y^2\left(\frac{2T\omega}{3\pi . B . \sin I}\right)^2=1.$$

This is the equation to an ellipse, whose axes are in the ratio of $\cos I : 1$. Thus is explained the construction in Woodhouse's *Astronomy*, new edition, p. 367.

33. PROP. 12. To investigate the motion of the pole produced by the Moon in one sidereal revolution.

By (9), it appears, that instead of considering at once the effect of the Sun and Moon upon the Earth, we may first investigate the effect produced by one, and then add to it the effect produced by the other. For the effect produced by the Moon, the investigation is exactly similar to those of Prop. 11 and 12, and the same figure may be used; observing that, as EC is the great circle apparently described by the Moon, Q is not now the pole of the ecliptic, but the pole of the Moon's orbit. There is only one difference: putting E for the Earth's absolute force, M for the Moon's, T' for the time of a sidereal revolution of the Moon, I' for the inclination of the Earth's axis to the axis of the Moon's orbit, a' for the mean distance, $\frac{dt}{dl}$ will be $\frac{r^2}{\sqrt{a'(1-e'^2)(E+M)}}$, and T' will be $\frac{2\pi . a'^{\frac{3}{2}}}{\sqrt{(E+M)}}$;

and, therefore, instead of $\frac{3\pi B}{T\omega}$, we must put $\frac{3\pi B}{T'\omega} \cdot \frac{M}{E+M}$.

If the Moon's mass be $\frac{1}{n}$th of the Earth's, this $= \frac{3\pi B}{T'\omega(n+1)}$.

Thus we find for the motion of the pole parallel to the Moon's orbit,

$$\frac{3\pi . B \sin I' . \cos I'}{T'\omega(n+1)} \left(C + l - \frac{\sin 2l}{2}\right):$$

and for the motion perpendicular to the Moon's orbit,

$$\frac{3\pi B . \sin I'}{T'\omega(n+1)} \cdot \frac{\cos 2l}{2},$$

l being now measured from the intersection of the equator with the Moon's orbit.

34. The latter expression, and the second term of the former, are periodical terms, going through all their changes of value twice in a month: their magnitudes, besides, are so small, that they are generally neglected.

Supposing l increased by 2π, we find, for the motion of the pole produced by the Moon's action in a sidereal revolution,

$$\frac{6\pi^2 . B . \sin I' . \cos I'}{T'\omega . (n+1)};$$

[which] motion is parallel to the Moon's orbit, or perpendicular [to] the great circle joining the pole of the Earth with the pole [of] the Moon's orbit.

35. PROP. 13. To investigate the precessional motion produced by the Moon's action during a long period.

Let Q, fig. 8, be the pole of the ecliptic; q that of the Moon's orbit; P that of the Earth: let them be joined by arcs of great circles; then, by the last article, it appears that by

the action of the Moon, the pole is in the time T' carried in the direction Pp, perpendicular to Pq, through the arc

$$\frac{6\pi^2 . B}{T'\omega(n+1)} \sin qP . \cos qP.$$

This may be represented by supposing the pole to have the velocity

$$\frac{6\pi^2 . B}{T'^2\omega(n+1)} \sin qP . \cos qP,$$

in the direction Pp. Its velocity then in a direction perpendicular to QP, is

$$\frac{6\pi^2 . B}{T'^2 . \omega(n+1)} \sin qP . \cos qP . \cos QPq.$$

This we must express in terms of QP, Qq, and the angle PQq.

36. Now,

$$\sin qP . \cos QPq = \frac{\cos Qq - \cos QP . \cos qP}{\sin QP};$$

but $\cos qP = \cos QP . \cos Qq + \sin QP . \sin Qq . \cos Q$;

substituting this,

$\sin qP . \cos QPq = \sin QP . \cos Qq - \cos QP . \sin Qq . \cos$

Multiplying by

$$\cos qP = \cos QP . \cos Qq + \sin QP . \sin Qq . \cos Q,$$

we find $\sin qP . \cos qP . \cos QPq$

$$= \sin QP . \cos QP . \cos^2 Qq - (\cos^2 QP - \sin^2 QP) . \sin Qq . \cos Qq . c$$
$$- \sin QP . \cos QP . \sin^2 Qq . \cos^2 Q.$$

Put I and i for QP and Qq: I and i are nearly consta

and, since the Moon's nodes revolve in a retrograde direction through a great circle in 18,6 years, if $\tau = 18{,}6$ years,

$$Q = \frac{2\pi t}{\tau}.$$

Hence, the velocity of the pole perpendicular to QP, is

$$\frac{6\pi^2 B}{T'^2 . \omega (n+1)} . \left\{ \sin I \cos I . \cos^2 i - \tfrac{1}{2} \cos 2I . \sin 2i . \cos \frac{2\pi t}{\tau} \right.$$

$$\left. - \sin I . \cos I . \sin^2 i . \cos^2 \frac{2\pi t}{\tau} \right\}$$

$$= \frac{6\pi^2 B}{T'^2 \omega (n+1)} \left\{ \sin I \cos I (\cos^2 i - \tfrac{1}{2} \sin^2 i) \right.$$

$$\left. - \tfrac{1}{2} \cos 2I . \sin 2i . \cos \frac{2\pi t}{\tau} - \tfrac{1}{2} \sin I . \cos I . \sin^2 i . \cos \frac{4\pi t}{\tau} \right\}.$$

Integrating this with respect to t, we find the precessional motion of the pole

$$= \frac{6\pi^2 B}{T'^2 \omega (n+1)} \left\{ \sin I \cos I (\cos^2 i - \tfrac{1}{2} \sin^2 i) . t \right.$$

$$- \frac{\tau}{4\pi} \cos 2I . \sin 2i . \sin \frac{2\pi t}{\tau}$$

$$\left. - \frac{\tau}{8\pi} \sin I . \cos I . \sin^2 i . \sin \frac{4\pi t}{\tau} \right\} + C.$$

37. The first term of this expression is proportional to the time, and, therefore, increases uniformly. The second term is periodical; it depends upon $\sin \frac{2\pi t}{\tau}$, or $\sin Q$, or sin longitude of Moon's ascending node. This is a part of Lunar Nutation. The third term depends upon sin 2 long. Moon's ascending node; it is, therefore, a part of Nutation: but its numerical value is so small, that it is commonly neglected.

38. To obtain the lunar precession of the equinoxes, we must, as before, divide the last expression by sin I. Thus, we get

$$\frac{6\pi^2 B}{T''^2 \omega (n+1)} \left\{ \cos I (\cos^2 i - \tfrac{1}{2} \sin^2 i) . t \right.$$

$$\left. - \frac{\tau}{4\pi} . \frac{\cos 2I}{\sin I} . \sin 2i . \sin \frac{2\pi t}{\tau} - \frac{\tau}{8\pi} \cos I . \sin^2 i . \frac{\sin 4\pi t}{\tau} \right\}.$$

The first term, which increases uniformly, is called simply the lunar precession; the second is the lunar equation of the equinoxes in longitude; the third is neglected. The lunar precession for a year is found by putting T for t; it is, therefore,

$$\frac{6\pi^2 B}{T''^2 \omega (n+1)} . \cos I (\cos^2 i - \tfrac{1}{2} \sin^2 i) . T.$$

39. Prop. 14. To investigate the alteration in the obliquity of the ecliptic produced by the Moon's action.

By (35), the velocity of the pole in the direction Pp, is

$$\frac{6\pi^2 . B}{T''^2 \omega (n+1)} \sin qP . \cos qP.$$

The velocity, therefore, with which the inclination is increased, is

$$\frac{6\pi^2 . B}{T''^2 \omega (n+1)} . \sin qP . \cos qP . \sin QPq.$$

$$\text{Now, } \sin qP . \sin QPq = \sin i . \sin Q;$$

$$\text{and } \cos qP = \cos I . \cos i + \sin I . \sin i . \cos Q.$$

Their product

$$= \cos I . \sin i . \cos i . \sin Q + \sin I . \sin^2 i . \sin Q . \cos Q;$$

or, the velocity of increase of inclination

$$\frac{6\pi^2 . B}{T'^2 \omega (n+1)} \times \left\{\frac{1}{2}\cos I . \sin 2i . \sin\frac{2\pi t}{\tau} + \frac{1}{4}\sin I . \sin^2 i . \sin\frac{4\pi t}{\tau}\right\}.$$

Integrating this with respect to t, the inclination

$$= I - \frac{6\pi^2 . B}{T'^2 \omega (n+1)} \times \left\{\frac{\tau}{4\pi}\cos I . \sin 2i . \cos\frac{2\pi t}{\tau} + \frac{\tau}{8\pi} . \sin I . \sin^2 i . \cos\frac{4\pi t}{\tau}\right\}.$$

The terms subtracted from I are periodical, depending upon $\cos\frac{2\pi t}{\tau}$, or cos long. Moon's ascending node, and upon the cosine of twice that angle; the first is a part of Lunar Nutation; the second is usually neglected.

40. If we call x and y the first terms of nutation in (36) and (39), it will easily be seen that they are connected by this equation,

$$x^2\left(\frac{2T'^2\omega(n+1)}{3\pi . B\tau . \cos 2I . \sin 2i}\right)^2 + y^2\left(\frac{2T'^2\omega(n+1)}{3\pi B . \tau . \cos I . \sin 2i}\right)^2 = 1.$$

This is the equation to an ellipse, in which the axes have the ratio of $\cos 2I : \cos I$. This explains the construction in Woodhouse's *Astronomy*, page 357.

41. In the two last Propositions we have considered i to be constant, and Q to be proportional to t. It appears, however, from Art. 68. of the *Physical Astronomy*, that the inclination is expressed, nearly, by

$$i . \left(1 + \frac{3m}{8}\cos 2 . \overline{\text{long. node} - \text{long. Sun}}\right);$$

and that from the longitude of the node, found on the supposition of its uniform retrogradation, we must subtract

$$\frac{3\,m}{8}\sin 2\,.\,\overline{\text{long. node} - \text{long. Sun}}.$$

Now, the Sun's longitude $= \dfrac{2\,\pi\,t}{T} + c$, nearly: the longitude of the node $= 360 - \dfrac{2\,\pi\,t}{\tau}$: hence, for i we ought to put

$$i\left\{1 + \frac{3\,m}{8}\cos 4\,\pi\,t\left(\frac{1}{T} + \frac{1}{\tau} + C\right)\right\},$$

and for Q we should put

$$\frac{2\,\pi\,t}{\tau} - \frac{3\,m}{8}\sin 4\,\pi\,t\left(\frac{1}{T} + \frac{1}{\tau} + C\right),$$

before performing the integrations: and $\sin 2\,Q$, &c. could be expanded, as in Art. 49, of *Physical Astronomy*. But the additional terms thus introduced have small coefficients, and upon integration receive large divisors, so that they become quite insensible. The expressions which we have found are, therefore, subject to no sensible error.

42. If now, in the terms of Nutation, which depend on twice the longitude of the Moon's ascending node, we used not the mean longitude of the node but the true, we should add to the expressions terms which have small coefficients, but which are not integrated, and, therefore, do not receive large divisors. The values thus found for the parts of Nutation at any given time would, therefore, sensibly differ from those found by the formulæ above. And, since the latter differ from the true ones only by quantities which are insensible, it follows that the values found by using the true longitude of the node, are sensibly erroneous. In calculating Nutation, therefore, the mean longitude of the node must be used, not the true.

43. Prop. 15. Assuming the law of density of the Earth, and the mass of the Moon, to calculate numerically the annual precession and the coefficients of solar and lunar nutation.

It is first necessary to calculate the value of B, or $\frac{\psi(c)}{\sigma(c)}$, which enters into all the expressions. Suppose, then, as in Prop. 29, of the Figure of the Earth, $\rho = A \cdot \frac{\sin qc}{c}$. By Art. 61, of the same,

$$\psi(c) = \frac{5}{3}\left(e - \frac{m}{2}\right) c^2 \cdot \phi(c);$$

$$\therefore\ B = \frac{5}{3}\left(e - \frac{m}{2}\right) c^2 \cdot \frac{\phi(c)}{\sigma(c)}.$$

$$\text{Now}\quad \phi(c) = \int_c \rho \frac{d \cdot c^3}{dc} = 3\int_c A \cdot c \cdot \sin qc$$

$$= 3A\left(-\frac{c \cdot \cos qc}{q} + \frac{\sin qc}{q^2}\right);$$

which, taken from $c = 0$ to $c = \mathrm{c}$, gives

$$\phi(\mathrm{c}) = 3A\left(-\frac{\mathrm{c} \cdot \cos q\mathrm{c}}{q} + \frac{\sin q\mathrm{c}}{q^2}\right).$$

$$\text{And}\quad \sigma(c) = \int_c \rho \cdot \frac{d \cdot c^5}{dc} \text{ by (26) } = 5A\int_c c^3 \sin qc$$

$$= 5A\left(\frac{-c^3 \cdot \cos qc}{q} + \frac{3c^2 \cdot \sin qc}{q^2} + \frac{6c \cdot \cos qc}{q^3} - \frac{6 \sin qc}{q^4}\right);$$

putting c for c, we have the value of σ(c). Hence

$$B = \frac{5}{3}\left(e - \frac{m}{2}\right) \times \frac{-q^3\mathrm{c}^3 \cdot \cos q\mathrm{c} + q^2\mathrm{c}^2 \cdot \sin q\mathrm{c}}{-q^3\mathrm{c}^3 \cdot \cos q\mathrm{c} + 3q^2\mathrm{c}^2 \sin q\mathrm{c} + 6q\mathrm{c} \cdot \cos q\mathrm{c} - 6 \sin q\mathrm{c}}$$

$$= \left(e - \frac{m}{2}\right) \times \frac{z}{2 + z - \frac{6z}{q^2\mathrm{c}^2}}, \text{ where } z = 1 - \frac{q\mathrm{c}}{\tan q\mathrm{c}}.$$

If we suppose q to have the value used in Art. 66, of the Figure of the Earth, namely, $\frac{5}{6} \cdot \frac{\pi}{c}$, and take the value of $e - \frac{m}{2}$, found in Art. 66, we get for the value of B, ,0031677.

44. The solar annual precession, which is found by supposing l to be increased by 2π in the expression of (30), is

$$\frac{6\pi^2 . B . \cos I}{T\omega}.$$

Now $T\omega =$ the angle described by the diurnal revolution of the Earth in 1 year $= 2\pi \times 366{,}26$; $I = 23^0 . 28'$: hence, the solar annual precession

$$= \frac{B \times 3\pi \times \cos 23^0 . 28'}{366{,}26}.$$

This is in parts of the radius: to reduce it to seconds we must multiply it by $\frac{180 \times 60 \times 60}{\pi}$. Thus the solar annual precession in seconds

$$= \frac{B \times 9 \times \overline{60}|^3 \times \cos 23^0 . 28'}{366{,}26}:$$

which, putting for B the value above, $= 15''$, 42.

45. From (30) it appears that the coefficient of $\sin 2l$, in the Solar Equation of the equinoxes in longitude, is to the annual solar precession as $\frac{1}{2} : 2\pi$. This coefficient, therefore,

$$= \frac{15'',42}{4\pi} = 1'',23.$$

And from (31), it appears that the coefficient of $\cos 2l$, in the Solar Nutation in obliquity, is to the annual solar precession as

$$\frac{\sin I}{2} : 2\pi . \cos I;$$

whence this coefficient

$$= \frac{15'',42}{4\,\pi \cot I} = 0'',33.$$

46. By (38), the lunar annual precession

$$= B \times \frac{6\,\pi^2}{T''\omega} \cdot \frac{T}{T'} \cdot \frac{\cos I}{n+1} \cdot \left(1 - \frac{3}{2}\sin^2 i\right).$$

Now, $T''\omega$ = angle described by the diurnal revolution of the Earth in one revolution of the Moon

$$= 2\,\pi \times 27,32 \;:\; \frac{T}{T'} = \frac{366,26}{27,32} \;:\; i = 5^0.8'.60'';$$

suppose the Moon's mass $\frac{1}{70}$ that of the Earth, or $n = 70$; hence, the lunar annual precession

$$= B \times \frac{3\,\pi \times 366,26 \times \cos 23^0.28' \times ,98791}{(27,32)^2 \times 71}.$$

This reduced to seconds, as the former was

$$= B \times \frac{9 \times \overline{60}|^3 \times 366,26 \times \cos 23^0.28' \times ,98791}{(27,32)^2 \times 71};$$

Putting the same value for B, this $= 38'', 57$.

By (38), the coefficient of $\sin \frac{2\,\pi t}{\tau}$, or sin long. Moon's ascending node, in the lunar equation of the equinoxes in longitude, is

$$B \times \frac{3\,\pi\,.\,\tau\,.\,\cos 2\,I\,.\,\sin 2\,i}{2\,T''^2.\,\omega\,(n+1)\,.\sin I}.$$

$$\text{Now } \frac{\tau}{T'^2\omega} = \frac{\tau}{T} \cdot \frac{T}{T'} \cdot \frac{1}{T''\omega} = \frac{18,6 \times 366,26}{(27,32)^2 \times 2\pi};$$

hence, this coefficient

$$= B \times \frac{3 \times 18,6 \times 366,26 \times \cos 2I \sin 2i}{4\,(27,32)^2 .\, 71 .\, \sin I},$$

and in seconds

$$= B \times \frac{9 \times 60^3 \times 18,6 \times 366,26 \times \cos 46^0.\,56' .\, \sin 10^0.\,17'.\,40''}{4\pi\,(27,32)^2 \;\; 71 .\, \sin 23^0.\,28'}.$$

Giving B the same value as before, the number of seconds is 19,3. And comparing (38) with (39), it is seen that the coefficient just found is to the coefficient of cos long. ascending node in the lunar nutation in obliquity, as

$$\frac{\cos 2I}{\sin I} : \cos I, \text{ or as } 1 : \tfrac{1}{2}\tan 2I:$$

whence this coefficient in seconds

$$= \frac{\tan 46^0.\,56' \times 19,3}{2} = 10,33.$$

The observed values of these coefficients are 18'',036 and 9'',6.

48. Adding together the numbers found in (44) and (45), the whole annual precession = 53'',99. The observed value is 50'',1.

49. Prop. 16. From observations on precession and nutation, to determine the Moon's mass, and the ellipticity of the Earth.

Among the various results of observation we shall select, as those most accurately determined, the whole annual precession in seconds (a), and the coefficient of the lunar nutation i

obliquity (b). Comparing the expressions in (38) and (39), the lunar annual precession

$$= b \times \frac{4\pi\left(1 - \frac{3}{2}\sin^2 i\right)}{\sin 2i} \cdot \frac{T}{\tau} = b \times C, \text{ suppose.}$$

Subtracting this from the whole annual precession, the solar annual precession $= a - bC$, and

$$\frac{\text{solar annual precession}}{\text{lunar annual precession}} = \frac{a - bC}{bC} = \frac{a}{bC} - 1.$$

But, by the expressions in (30) and (38),

$$\frac{\text{solar annual precession}}{\text{lunar annual precession}} = \frac{T'^2\,(n+1)}{T^2\left(1 - \frac{3}{2}\sin^2 i\right)}:$$

$$\text{hence } \frac{a}{bC} - 1 = \frac{T'^2\,(n+1)}{T^2\left(1 - \frac{3}{2}\sin^2 i\right)} = D\,(n+1), \text{ suppose;}$$

$$\text{and } n + 1 = \frac{1}{D}\left(\frac{a}{bC} - 1\right).$$

This determines n, that is, the ratio of the mass of the Earth to the mass of the Moon. If we calculate the values of C and D with the values of i, $\frac{T'}{T}$, and $\frac{\tau}{T}$, already given,

$$n + 1 = \frac{a}{b} \times 47{,}52 - 177{,}56.$$

50. The ellipticity of the Earth cannot be determined at once from these data, but an equation can be found depending on the law of density of its strata, such that if a law be assumed

with one indeterminate coefficient, that coefficient can be found by approximation, and the ellipticity of the Earth can then be determined. The solar annual precession we have found $= a - b \,.\, C$: but the expression for it in seconds, by (30), is

$$B \times \frac{9 \times 60^3 \times \cos 23^0 . 28'}{366,26} = B \times 4869;$$

making these equal, and putting for C its value,

$$B = a \times ,0002054 - b \times ,0007675.$$

This gives the value of

$$\frac{5}{2}\left(e - \frac{m}{2}\right) . c^2 . \frac{\phi\,(c)}{\sigma\,(c)}.$$

If, as in Art. 65, of the Figure of the Earth, we assume for the form of the expression for ρ, $A\,\dfrac{\sin qc}{c}$, we have this equation,

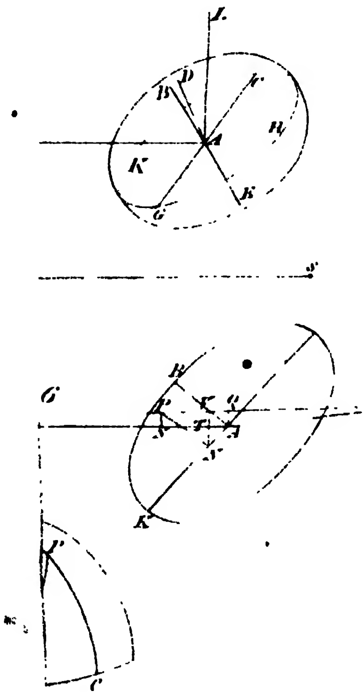
$$B = \left\{ \frac{5m}{2} . \frac{\dfrac{3z}{q^2c^2} - 1}{\dfrac{q^2c^2}{z} + z - 3} - \frac{m}{2} \right\} . \frac{z}{2 + z - \dfrac{6z}{q^2c^2}},$$

$$\text{where } z = 1 - \frac{qc}{\tan qc}:$$

finding the value of qc by approximation, the ellipticity of the surface will be found by substituting it in the equation

$$e = \frac{5m}{2} . \frac{\dfrac{3z}{q^2c^2} - 1}{\dfrac{q^2c^2}{z} + z - 3}.$$

51. The values of a and b, according to the observations of most astronomers, are $50'',1$ and $9'',6$. Substituting these



with one indeterminate coefficient, that coefficient can be found by approximation, and the ellipticity of the Earth can then be

51. The values of a and b, according to the observations of most astronomers, are $50'',1$ and $9'',6$. Substituting these in the formulæ above, $n = 68,9$, and $B = ,0029225$. Assuming the law of density mentioned above, $qc = 160^0 . 53'$, and

$$e = ,0030812 = \frac{1}{324,5}.$$

52. Dr. Brinkley, in the *Philosophical Transactions* for 1821, has reduced the value of b to $9'',25$. This gives $n = 78,2$, and $B = ,0031911$; thence $qc = 149^0 . 10'$, and

$$e = ,0032973 = \frac{1}{303,3}$$

CALCULUS OF VARIATIONS.

1. In solving problems of maxima and minima by the Differential Calculus, it is necessary to express the quantity, which is to be made a maximum or minimum, in terms of the independent variable, or at least to find an equation between them; in all cases it is necessary to assign a relation between the function (u) and the independent variable (x), so that for any given value of x, the corresponding value of u actually can be found. When this is the case, $\frac{du}{dx}$ can be found; and if we make it $= 0$, an equation is obtained, which, combined if necessary with the original equation, determines the value of x or u, corresponding to the maximum or minimum value of u; and the problem is solved.

2. But it is sometimes necessary to solve problems of maxima and minima, when the relation between u and x cannot be expressed, u depending generally upon an integral $\int_x V$, where V involves y and its differential coefficients, and where it is the *object* of the problem to find the *relation* between y and x. Suppose, for instance, it were required to find the curve of quickest descent from one given point to anothe[illegible] x be measured horizontally from the first point, and y ve[illegible] and if the time $= u$, we have

$$\frac{du}{dx} = \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}}, \text{ or } u = \int_x \frac{\sqrt{1+p^2}}{\sqrt{2gy}}$$

where $p = \frac{dy}{dx}$.

This expression plainly cannot be integrated except the relation of y to x be given; but this it is the object of the problem to discover. Here then the methods of the Differential Calculus entirely fail: and some new process must be devised for the solution of problems similar to the preceding.

3. The method given by Lagrange bears a close analogy to the methods of the Differential Calculus. In order to find what must be the relation between x and y, to make u a maximum or minimum, we must conceive an expression to be assumed for y, and this expression to be then altered by the quantity δy, δy being some function of x. Now, in order that the assumed value of y may possess the desired property of making u a maximum or minimum, it is necessary that, upon substituting $y + \delta y$ and $y - \delta y$ for y, u may, by both substitutions, be increased, or by both be diminished. From this it follows, by reasoning precisely similar to that employed in the Differential Calculus, that the sum of the terms depending on the first power of δy, in the new values of u, must $= 0$. If then we can by any means express this sum, we shall be able to find the relation between x and y, that will make u a maximum or minimum.

4. This method is sufficient when, supposing u expressed by an integral, the values of x at the limits are known: as, for instance, in the problem of (2), the extreme points of the curve are given. But these may be undetermined, as in the problem "To find the line of quickest descent from one given curve to [illegible] the given curve." Suppose Aa, Bb, fig. 1, to be the [illegible] rves; APB the curve required. If we suppose δy to [illegible] h a form that for the values of x, corresponding to the [illegible] and B, δy is $= 0$, the curve APB will be changed by [illegible] on to such a curve as ApB. But though the consideration [illegible] the effect of this variation will assist us in discovering the curve APB, which is the line of shortest descent from the point A, to the point B, it plainly will not enable us to determine what points of the given curves must be selected for the extremities of the curve required. If we suppose the form of

$\delta y'$ to be such, that δy is not $=0$ for the values of x corresponding to the points A and B, this variation will change the curve APB into $A'P'B'$, in which the values of x for A' and B', are the same as those for A and B. Since this new curve is not terminated by the given curves, the variation is not such as the conditions of the problem require. The method which we have given, is, therefore, defective; its deficiency is supplied in the following manner.

5. Instead of supposing the curve to be varied by the variation of only one of its co-ordinates (y), suppose both co-ordinates to be varied; that is, suppose the co-ordinates of the new curve to be

$$x' = x + \delta x, \quad \text{and} \quad y' = y + \delta y,$$

δx and δy being functions of x. This amounts to supposing, that in fig. 2, x and y being the co-ordinates of P,

$$NN' = \delta x, \quad \text{and} \quad QP' = \delta y;$$

and that, by taking successively other points of AB, a series of points is determined, through which $A'B'$ is drawn. Now, if we take care to give δx and δy, at the extremities of AB, such a relation, that the point A' will be found in the curve Aa, and the point B' in the curve Bb, the variation will be such, as the conditions of the problem require. And as the form of the function which we suppose to represent δx or δy generally, is absolutely arbitrary, subjected to no other condition, than that δx and δy at the limits, shall have a given ratio; this variation is the most general, which it is possible for us to give.

6. That the value of u may be a maximum or [illegible] the sum of the terms depending on the first powers of [illegible] δy, as is shown by the same kind of reasoning as [illegible] used in the Differential Calculus, must $=0$. In finding the variation of u, therefore, we may confine ourselves to the terms depending on the first power of δx and δy.

7. For this purpose, we will find δp, δq, &c.

$$\left(p = \frac{dy}{dx},\ q = \frac{d^2 y}{dx^2},\ \&c.\right),$$

as far as the first power of δx and δy. The original value of p was $\frac{dy}{dx}$; the value, after giving to x and y a variation, is $\frac{dy'}{dx'}$;

$$\text{hence, } \delta p = \frac{dy'}{dx'} - \frac{dy}{dx}.$$

$$\text{Now } \frac{dy'}{dx'} = \frac{\frac{dy'}{dx}}{\frac{dx'}{dx}} = \frac{\frac{dy}{dx} + \frac{d\,.\,\delta y}{dx}}{1 + \frac{d\,.\,\delta x}{dx}}$$

$$= \frac{dy}{dx} + \frac{d\,.\,\delta y}{dx} - \frac{dy}{dx}\,.\,\frac{d\,.\,\delta x}{dx};$$

$$\therefore\ \delta p = \frac{d\,.\,\delta y}{dx} - p\,.\,\frac{d\,.\,\delta x}{dx}.$$

$$\text{Similarly, } \delta q = \frac{dp'}{dx'} - \frac{dp}{dx} = \frac{\frac{dp'}{dx}}{\frac{dx'}{dx}} - \frac{dp}{dx}$$

$$= \frac{\frac{dp}{dx} + \frac{d\,.\,\delta p}{dx}}{1 + \frac{d\,.\,\delta x}{dx}} - \frac{dp}{dx} = \frac{d\,.\,\delta p}{dx} - q\frac{d\,.\,\delta x}{dx}.$$

$$\text{And } \delta r = \frac{d\,.\,\delta q}{dx} - r\frac{d\,.\,\delta x}{dx},\ \&c.$$

Let $u = \int_x V$; then $\delta u = \int_{x'} V' - \int_x V$,

being the same function of x', y', &c. that V is of x, y, &c.)

$$=\int_x V'\frac{dx'}{dx}-\int_x V=\int_x\left(V'\frac{dx'}{dx}-V\right).$$

Now, suppose V to be a function of x, y, p, q, &c., and let

$$\frac{dV}{dx}=M,\ \frac{dV}{dy}=N,\ \frac{dV}{dp}=P,\ \frac{dV}{dq}=Q, \text{ \&c.},$$

then, (to the first power of δx, &c.)

$$V'=V+M\delta x+N\delta y+P\delta p+Q\delta q+\text{\&c.},$$

$$\text{and } \frac{dx'}{dx}=1+\frac{d.\delta x}{dx},$$

$$\therefore\ V'\frac{dx'}{dx}-V=V.\frac{d.\delta x}{dx}+M\delta x+N\delta y+P\delta p+Q\delta q+\text{\&c.}$$

Hence, δu, (integrating the first term by parts)

$$=V\delta x-\int_x\delta x.\frac{d(V)}{dx}+\int_x(M\delta x+N\delta y+P\delta p+Q\delta q+\text{\&c.}).$$

The integrations are performed with respect to x, considering y, p, q, &c. as functions of x; hence, for $\frac{d(V)}{dx}$ we must put

$$\frac{dV}{dx}+\frac{dV}{dy}.\frac{dy}{dx}+\frac{dV}{dp}.\frac{dp}{dx}+\frac{dV}{dq}.\frac{dq}{dx}+\text{\&c.}$$

$$=M+Np+Pq+Qr+\text{\&c.}$$

$$\text{then, } \delta u=V\delta x+\int_x\{N(\delta y-p\delta x)$$

$$+P(\delta p-q\delta x)+Q(\delta q-r\delta x)+\text{\&c.}\}$$

9. Now, upon substituting the values found above for δp, δq, &c., we find

$$\delta p-q\delta x=\frac{d.\delta y}{dx}-p\frac{d.\delta x}{dx}-q\delta x=\frac{d(\delta y-p\delta x)}{dx};$$

$$\delta q - r\delta x = \frac{d\,.\,\delta p}{dx} - q\,\frac{d\,.\,\delta x}{dx} - r\delta x$$

$$= \frac{d(\delta p - q\delta x)}{dx} = \frac{d^2(\delta y - p\delta x)}{dx^2};$$

and so on. Let $\delta y - p\delta x = \omega$; then $\delta u =$

$$V\delta x + \int_x (N\omega + P\frac{d\omega}{dx} + Q\frac{d^2\omega}{dx^2} + \&c.).$$

Integrating by parts,

$$\int_x P\frac{d\omega}{dx} = P\omega - \int_x \omega\frac{d(P)}{dx};$$

$$\int_x Q\frac{d^2\omega}{dx^2} = Q\frac{d\omega}{dx} - \frac{d(Q)}{dx}\omega + \int_x \omega\frac{d^2(Q)}{dx};$$

and so for the others: it being observed, that $\frac{d(P)}{dx}$, &c. signify the differential coefficients with respect to x, considering y, p, q, &c. as functions of x. Hence, we have δu

$$= V\delta x + \omega\{P - \frac{d(Q)}{dx} + \&c.\}$$

$$+ \frac{d\omega}{dx}(Q - \&c.)$$

$$+ \&c.$$

$$+ \int_x \omega\{N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c.\}.$$

Putting for ω, $\frac{d\omega}{dx}$, &c. their values $\delta y - p\delta x$, $dp - q\delta x$,

$$\delta u = V\delta x + (\delta y - p\delta x)\,.\,\{P - \frac{d(Q)}{dx} + \&c.\}$$

$$+ (\delta p - q\delta x)\,.\,\{Q - \&c.\}$$

$$+ \&c.$$

$$\int_x \omega\{N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c.\}.$$

Or, since this quantity results from integration, and must, therefore, be taken between limits, if we put $x_{,}$, $y_{,}$, $p_{,}$, $V_{,}$, &c. for the values of x, y, p, V, &c., at the first limit, and $x_{,,}$, $y_{,,}$, $p_{,,}$, $V_{,,}$, &c. for those at the second limit, we have δu

$$
\begin{aligned}
&= V_{,,}\delta x_{,,} - V_{,}\delta x_{,} \\
&+ (\delta y_{,,} - p_{,,}\delta x_{,,}).\left\{P_{,,} - \frac{d(Q_{,,})}{dx_{,,}} + \&c.\right\} \\
&- (\delta y_{,} - p_{,}\delta x_{,})\left\{P_{,} - \frac{d(Q_{,})}{dx_{,}} + \&c.\right\} \\
&+ (\delta p_{,,} - q_{,,}\delta x_{,,})(Q_{,,} - \&c.) - (\delta p_{,} - q_{,}\delta x_{,})(Q_{,} - \&c.) \\
&+ \&c. \qquad\qquad\qquad - \&c. \\
&+ \int_{x}\omega\left\{N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c.\right\}.
\end{aligned}
$$

the last integral being supposed to be taken between the same limits as the others.

10. When u is a maximum or minimum, the expression above must $=0$. Now this expression consists of two parts perfectly different; the first involving only the values of δx and δy at the limits, the second being an integral dependent on the general values of δx and δy. Now it would be possible to assign different forms for δx and δy, which should leave their values at the limits unaltered, while the value of the integral

$$\int_{x}\omega\left\{N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c.\right\},$$

should be altered in any way whatever. In order then that this may $=0$, whatever be the form of δx and δy, we must make these two parts separately $=0$, which can be done only by making

$$N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c. = 0,$$

and $V_{,,}\delta x_{,,} - V_{,}\delta x_{,} + (\delta y_{,,} - p_{,,}\delta x_{,,})\left\{P_{,,} - \frac{d(Q_{,,})}{dx_{,,}} + \&c.\right\}$

$$\pm (\delta y_{,} - p_{,}\delta x_{,}) \{P_{,} - \frac{d(Q_{,})}{dx_{,}} + \&c.\} + \&c. = 0.$$

In the latter of these equations, we must eliminate, by means of given relations, as many as possible of the quantities $\delta x_{,}$, $\delta x_{,,}$, $\delta y_{,}$, &c., and make the coefficient of each remaining one separately equal to nothing.

11. The equation

$$N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c. = 0,$$

may, in most cases, be rendered more easy of application by integration. Thus, suppose there enter into V only y and p; the equation is reduced to

$$N - \frac{d(P)}{dx} = 0, \text{ or } Np = p \, . \frac{d(P)}{dx};$$

$$\therefore \frac{d(V)}{dx} = N\frac{dy}{dx} + P\frac{dp}{dx} = p \, . \frac{d(P)}{dx} + P\frac{dp}{dx};$$

integrating, $V = Pp + C$. Suppose V involved only p and q;

$$\text{then } \frac{d(P)}{dx} = \frac{d^2(Q)}{dx}, \quad P = \frac{d(Q)}{dx} + C;$$

$$\frac{d(V)}{dx} = P\frac{dp}{dx} + Q\frac{dq}{dx} = q\frac{d(Q)}{dx} + Cq + Q\frac{dq}{dx};$$

integrating, $V = Qq + Cp + C'$.

V contained only y and q; then

$$N + \frac{d^2(Q)}{dx^2} = 0, \quad \text{or } N = -\frac{d^2(Q)}{dx^2};$$

$$\therefore N\frac{dy}{dx} + Q\frac{dq}{dx} = -p\frac{d^2(Q)}{dx^2} + Q\frac{dq}{dx};$$

$$\text{integrating, } V = Qq - p\frac{d(Q)}{dx} + C.$$

And similarly in other cases.

12. We proceed to illustrate, by examples, the application of these formulæ.

It is required to find the shortest line that can be drawn from one given point, to another given point. Here

$$u = \int_x \sqrt{1+p^2}; \quad V = \sqrt{1+p^2},$$

into which p only enters; the equation

$$N - \frac{d(P)}{dx}, \text{ \&c.} = 0, \quad \text{becomes } \frac{d(P)}{dx} = 0,$$

$$P = C, \quad \text{or } \frac{p}{\sqrt{1+p^2}} = C; \quad \therefore p = \frac{C}{\sqrt{1-C^2}} = C';$$

and it is a straight line. Since the extremities are invariable $\delta x_{,}$, $\delta x_{,,}$, $\delta y_{,}$, $\delta y_{,,}$, are all $=0$; and, therefore, the first part o δu vanishes without any relation between V, P, &c.

13. It is required to find the shortest line that can be drawn from one given curve to another given curve. Here

$$u = \int_x \sqrt{1+p^2}, \quad V = \sqrt{1+p^2},$$

and, as before,

$$\frac{d(P)}{dx} = 0, \quad P \text{ or } \frac{p}{\sqrt{1+p^2}} = C, \quad p = \frac{C}{\sqrt{1-C^2}}$$

The other equation becomes

$$V_{,,}\delta x_{,,} - V_{,}\delta x_{,} + P_{,,}(\delta y_{,,} - p_{,,}\delta x_{,,}) - P_{,}(\delta y_{,} - p_{,}\delta x_{,}) = 0,$$

$$\text{or } P_{,,}\delta y_{,,} + (V_{,,} - P_{,,}p_{,,})\delta x_{,,} - P_{,}\delta y_{,} - (V_{,} - P_{,}p_{,})\delta x_{,} = 0.$$

Let AB, (fig. 2), be the line required; then, in changing A to $A'B'$, as the point A' is to be found in the curve Aa, th

ratio of $\delta x_{,}$ to $\delta y_{,}$ or of At to tA', must be the same as the ratio of the increment of x, to the increment of y, in the curve Aa. Let $\frac{dy}{dx} = m$, be the differential equation of the curve Aa; then $\delta y_{,}$ must $= m\delta x_{,}$. In the same manner, if $\frac{dy}{dx} = n$ be the differential equation of the curve Bb, $\delta y_{,,} = n\delta x_{,,}$. Also,

$$p_{,} = C'; \quad p_{,,} = C'';$$

$$V_{,} - P_{,}p_{,} = \sqrt{1+p_{,}^2} - \frac{p_{,}^2}{\sqrt{1+p_{,}^2}} = \frac{1}{\sqrt{1+p_{,}^2}} = \frac{1}{\sqrt{1+C'^2}};$$

$$V_{,,} - P_{,,}p_{,,} = \frac{1}{\sqrt{1+C''^2}};$$

$$\text{and } P_{,} = P_{,,} = \frac{C''}{\sqrt{1+C''^2}}.$$

Substituting, the equation becomes

$$\left(\frac{nC_{,}}{\sqrt{1+C_{,}^2}} + \frac{1}{\sqrt{1+C_{,}^2}}\right)\delta x_{,,} - \left(\frac{mC_{,}}{\sqrt{1+C_{,}^2}} + \frac{1}{\sqrt{1+C_{,}^2}}\right)\delta x_{,} = 0$$

and since [illegible]e cannot assign any relation which will, in all cases, subsist bet[illegible]een $\delta x_{,}$, and $\delta x_{,,}$, we must make each coefficient $= 0$,

$$\text{or, } 1 + nC' = 0, \quad 1 + mC' = 0.$$

[illegible]uations shew, that the line required must cut both the [illegible]ves at right angles.

[illegible]is required to find the curve of quickest descent from one give[illegible]oint, to another given point. Let the higher point be made [illegible] origin of co-ordinates.

Here [illegible] $\frac{\sqrt{1+p^2}}{\sqrt{2gy}}$; and, therefore, $u = \int_x \frac{\sqrt{1+p^2}}{\sqrt{y}}$

must be a minimum; $V = \frac{\sqrt{1+p^2}}{\sqrt{y}}$; and as this contains only y and p, we must have, by (11),

$$V = Pp + \frac{1}{C}.$$

$$\text{Now } P = \frac{p}{\sqrt{y}\sqrt{1+p^2}};$$

$$\therefore \frac{\sqrt{1+p^2}}{\sqrt{y}} = \frac{p^2}{\sqrt{y}\sqrt{1+p^2}} + \frac{1}{C},$$

$$\text{or } \sqrt{y}\,.\sqrt{1+p^2} = C;$$

$$\therefore \frac{dx}{dy} = \frac{1}{p} = \frac{y}{\sqrt{C^2 y - y^2}};$$

$$\text{integrating, } x + C' = \frac{C^2}{2}\text{ ver sin}^{-1}\frac{2y}{C^2} - \sqrt{C^2 y - y^2};$$

the equation to a cycloid, whose base is horizontal, whose vertex is downwards, and whose cusp is at the higher of the given points.

15. It is required to find the curve of quickest descent from one given curve to another given curve, the velocity at every point being that acquired by falling from a given horizontal line.

Here, (as in the last), $V = \frac{\sqrt{1+p^2}}{\sqrt{y}}$; and th

$$N - \frac{d(P)}{dx} + \&c. = 0,$$

$$\text{gives } \sqrt{y}\sqrt{1+p^2} = C.$$

Also, $V - Pp = \frac{1}{C}$; and $P = \frac{p}{\sqrt{y}\sqrt{1 \cdot}}$

Hence, the equation for the limits becomes

$$\frac{p_{\prime\prime}}{C}\delta y_{\prime\prime} + \frac{1}{C}\delta x_{\prime\prime} - \frac{p_{\prime}}{C}\delta y_{\prime} - \frac{1}{C}\delta x_{\prime} = 0;$$

or, if $\frac{dy}{dx} = m$, $\frac{dy}{dx} = n$, be the differential equations to the given curves, $\delta y_{\prime}$ must $= m\,\delta x_{\prime}$, and $\delta y_{\prime\prime}$ must $= n\,\delta x_{\prime\prime}$, and

$$\frac{np_{\prime\prime}+1}{C}\delta x_{\prime\prime} - \frac{mp_{\prime}+1}{C}\delta x_{\prime} = 0.$$

Since there is no relation between $\delta x_{\prime}$ and $\delta x_{\prime\prime}$, we must have

$$np_{\prime\prime}+1=0, \quad mp_{\prime}+1=0;$$

that is, the cycloid will cut both the curves at right angles. The cusp of the cycloid, as appears from the equation, will not be at the point at which the falling body leaves the first curve, but will be in the given horizontal line.

16. To find the form of a solid of revolution, that the resistance in moving through a fluid in the direction of its axis, on the usual suppositions, may be a minimum.

The resistance $\propto \int_x y\,\frac{p^3}{1+p^2}$;

$$\therefore V = \frac{yp^3}{1+p^2}; \quad P = y\,\frac{3p^2(1+p^2)-2p^4}{(1+p^2)^2} = y\cdot\frac{3p^2+p^4}{(1+p^2)^2};$$

and the equation $V = Pp + C$ gives

$$y\,\frac{p^3+p^5}{(1+p^2)^2} = y\,\frac{3p^3+p^5}{(1+p^2)^2} + C,$$

$$\text{or } \frac{2yp^3}{(1+p^2)^2} + C = 0,$$

the differential equation to the curve, by whose revolution the solid required is generated.

17. To find the curve, which, of all that can be drawn between two given points, contains between the curve, the

evolute, and the radii of curvature at the extremities, the least area. If h be the increment of x at any point, the corresponding increment of the arc is ultimately $h\sqrt{1+p^2}$, the radius of curvature $= \dfrac{(1+p^2)^{\frac{3}{2}}}{-q}$, hence, the increment of the area is ultimately $h\,\dfrac{(1+p^2)^2}{-2q}$, $\therefore\ u = \displaystyle\int_x \frac{(1+p^2)^2}{q}$ must be a minimum

$$\text{Here,}\quad V = \frac{(1+p^2)^2}{q},$$

and as this involves only p and q,

$$V = Qq + Cp + C', \text{ by (11)}$$

$$\text{And}\quad Q = -\frac{(1+p^2)^2}{q^2}; \quad \therefore \frac{2(1+p^2)^2}{q} = Cp + C'.$$

$$\text{Hence,}\quad \frac{(Cp+C')q}{(1+p^2)^2} = 2;$$

$$\text{integrating,}\quad \frac{-C + C'p}{1+p^2} + C'\tan^{-1}p = 4(x+a).$$

$$\text{And}\quad \frac{(Cp^2 + C'p)\,q}{(1+p^2)^2} = 2p;$$

$$\text{integrating,}\quad \frac{-C' - Cp}{1+p^2} + C\tan^{-1}p = 4(y+b).$$

Eliminating $\tan^{-1}p$,

$$\frac{C'^2 - C^2 + 2CC'.p}{1+p^2} = 4\{C(x+a) - C'(y+b)\}$$

$$\text{Let } C = 4f\sin\theta; \quad C' = 4f\cos\theta;$$

substituting,

$$f.\frac{\cos 2\theta + \sin 2\theta\ p}{1+p^2} = \sin\theta\,(x+a) - \text{co}\ [illegible]$$

To transform this into a more simple equation, we will first change the origin of co-ordinates, preserving their direction; if x' and y' be measured from a point whose co-ordinates are $-a$ and $-b$.

$$\text{then, } x' = x + a, \quad y' = y + b, \quad p' = p,$$

and the equation becomes

$$f.\frac{\cos 2\theta + \sin 2\theta . p'}{1 + p'^2} = \sin\theta . x' - \cos\theta . y'.$$

Now, take a new system of co-ordinates x'' and y'', (fig. 3.) having the same origin as x' and y', but inclined to them at an angle θ; then

$$y' = x''\sin\theta - y''\cos\theta; \quad x' = x''\cos\theta + y''\sin\theta;$$

$$p' = \frac{dy'}{dx'} = \frac{\frac{dy'}{dx''}}{\frac{dx'}{dx''}} = \frac{\sin\theta - \cos\theta . p''}{\cos\theta + \sin\theta . p''};$$

$$\text{and substituting, } f.\frac{\cos^2\theta - \sin^2\theta . p''^2}{1 + p''^2} = y'';$$

$$\therefore \frac{f}{1 + p''^2} = y'' + f\sin^2\theta, \quad \text{or } (y'' + f.\sin^2\theta)(1 + p''^2) = f,$$

the equation to a cycloid. If the position of the tangents at the [illegible] points be given, the constants must be determined, so [illegible]cloid pass through the given points, and touch [illegible]ven tangents. If the extreme points only be fixed, leaving [illegible]ctions of the tangents indeterminate, $\delta x_{,}$, $\delta y_{,}$, $\delta x_{,,}$, $\delta y_{,,}$, [illegible] and the equation

$$[illegible]\,\delta x_{,,} - V_{,}\delta x_{,} + (\delta y_{,,} - p_{,,}\delta x_{,,}).\left\{P_{,,} - \frac{d(Q_{,,})}{dx_{,,}}\right\}$$

$$- (\delta y_{,} - p_{,}\delta x_{,})\left\{P_{,} - \frac{d(Q_{,})}{dx_{,}}\right\}$$

$$+ (\delta p_{,,} - q_{,,}\delta x_{,,})\, Q_{,,} - (\delta p_{,} - q_{,}\delta x_{,})\, Q_{,} = 0,$$

is reduced to

$$Q_{,,}\delta p_{,,} - Q_{,}\delta p_{,} = 0.$$

Let $\phi_{,}$ and $\phi_{,,}$ be the angles which the directions of the tangents at the extreme points make with the axis of x; then

$$p_{,,} = \tan \phi_{,,}; \quad \delta p_{,,} = (1 + p_{,,}^2)\,\delta\phi_{,,}; \quad \delta p_{,} = (1 + p_{,}^2)\,\delta\phi_{,};$$

$$\text{also } Q_{,,} = \frac{-(1 + p_{,}^2)^2}{q_{,,}^2}, \quad Q_{,} = -\frac{-(1 + p_{,}^2)^2}{q_{,}^2};$$

and making equal to nothing the coefficients of $\delta\phi_{,,}$ and $\delta\phi_{,}$, we find

$$\left\{\frac{(1 + p_{,,}^2)^{\frac{3}{2}}}{q_{,,}}\right\}^2 = 0, \quad \left\{\frac{(1 + p_{,}^2)^{\frac{3}{2}}}{q_{,}}\right\}^2 = 0;$$

that is, at the extremities, the radii of curvature are each $= 0$; therefore these points are cusps; therefore the curve is a complete cycloid.

18. In all the examples above, we have supposed, as is commonly the case, that V does not involve the limits of the integral. But it may happen, that V will involve the values of x, y, p, &c. at the limits. In that case we must recur to the investigation in (8); instead of giving V' the value which it has there, we must put

$$V' = V + \frac{dV}{dx_{,}}\delta x_{,} + \frac{dV}{dy_{,}}\delta y_{,} + \text{\&c.}$$

$$+ M\delta x + N\delta y + P\delta p + \text{\&c.}$$

By going through the same operation as is there perfor[illegible] find δu

$$= V_{,,}\delta x_{,,} - V_{,}\delta x_{,}$$

$$+ (\delta y_{,,} - p_{,,}\delta x_{,,})\left\{P_{,,} - \frac{d(Q_{,,})}{dx_{,,}} + \text{\&c.}\right.$$

$$- (\delta y_{,} - p_{,}\delta x_{,}) \left\{P_{,} - \frac{d(Q)}{dx_{,}} + \&c.\right\}$$

$+ \&c.$

$$+\int_x \omega \left\{N - \frac{d(P)}{dx} + \frac{d^2(Q)}{dx^2} - \&c.\right\}$$

$$+\int_x \left(\frac{dV}{dx_{,}}\delta x_{,} + \frac{dV}{dy_{,}}\delta y_{,} + \&c.\right).$$

Now, $\delta x_{,}$, $\delta y_{,}$, &c., are not functions of x, generally, but of the limiting values of x; and, therefore, in the integration they may be considered constant with respect to x, and may be put before the integral sign as multipliers. The last line, therefore is not to be connected with that immediately preceding, but with the former parts. Thus we have δu

$$= V_{,,}\delta x_{,,} - V_{,}\delta x_{,} + \delta x_{,}\int_x \frac{dV}{dx_{,}} + \delta y_{,}\int_x \frac{dV}{dy_{,}} + \&c.$$

$$+ (\delta y_{,,} - p_{,,}\delta x_{,,}) . (P_{,,} - \&c.) - (\delta y_{,} - p_{,}\delta x_{,}) . (P_{,} - \&c.)$$

$+ \&c.$

$$+\int_x \omega \left\{N - \frac{d(P)}{dx} + \&c.\right\}.$$

19. Suppose, for instance, it were required to find the curve of quickest descent from one given curve to another given curve, [illegible] being supposed to commence at the first.

Here, $t = \int_x \frac{\sqrt{1+p^2}}{\sqrt{2g(y-y_{,})}}$;

$$\int_x \frac{\sqrt{1+p^2}}{\sqrt{y-y_{,}}}; \quad \text{then } V = \frac{\sqrt{1+p^2}}{\sqrt{y-y_{,}}};$$

$$P = \frac{p}{\sqrt{1+p^2}\sqrt{y-y_{,}}};$$

and by (11),

$$V = Pp + \frac{1}{C}, \text{ or } \sqrt{y - y_,}\,\sqrt{1 + p^2} = C,$$

and the curve is a cycloid in the same position as before, its cusp being at the point from which the motion begins. Now $x_,$, $x_{,,}$, $y_{,,}$, do not enter into V: and to integrate $\frac{dV}{dy_,}$, we observe that, from the form of V in this particular case,

$$\frac{dV}{dy_,} = -\frac{dV}{dy} = -N = -\frac{d(P)}{dx}\ \left(\text{since } N - \frac{d(P)}{dx} = 0\right);$$

$$\therefore \int_x \frac{dV}{dy_,} = -P:$$

which, taken between its limits, is $-P_{,,} + P_,$. Hence, we have for the first part of δu,

$$V_{,,}\delta x_{,,} + P_{,,}(\delta y_{,,} - p_{,,}\delta x_{,,}) - V_,\delta x_, - P_,(\delta y_, - p_,\delta x_,) + (P_, - P_{,,})\,\delta y_,$$

$$= (V_{,,} - P_{,,}p_{,,})\,\delta x_{,,} + P_{,,}\delta y_{,,} - (V_, - P_,p_,)\,\delta x_, - P_{,,}\delta y_,$$

$$= \frac{1}{C}\,\delta x_{,,} + \frac{p_{,,}}{C}\,\delta y_{,,} - \frac{1}{C}\,\delta x_, - \frac{p_{,,}}{C}\,\delta y_,.$$

And, if the equations to the limiting curves be

$$\frac{dy}{dx} = m, \quad \frac{dy}{dx} = n,$$

we must, as before, put $\delta y_, = m\delta x_,$, $\delta y_{,,} =$
find

$$(1 + p_{,,}n)\,\delta x_{,,} - (1 + p_{,,}m)\,\delta x_, = 0$$

then, since $\delta x_,$ and $\delta x_{,,}$ are indeterminate, we al

$$1 + p_{,,}n = 0,$$

which shews that the cycloid cuts the second curve at right angles, and

$$1 + p_{\prime\prime} m = 0,$$

which gives $m = n$, and shews that at the points where the cycloid meets both curves, their tangents are parallel.

20. There yet remains a very extensive class of problems: those in which the value of one function (v) is given, while another (u) is to be made a maximum or minimum. For instance; it is required to find the form of a curve, whose length is given, that the area contained by it may be the greatest possible. If we take the variation of u as in (9), we must not as in (10) make the two parts, of which it consists, separately $=0$; for it is not necessary that δu be $= 0$ for any values whatever of δx and δy, but only for such values as make $\delta v = 0$; a condition which would by that process be entirely neglected. If, however, we make $\delta u + a \delta v = 0$ (a being a constant to be determined), on the supposition that δx and δy have any values whatever, then, the values of δx and δy which make $\delta v = 0$, and no other, will make $\delta u = 0$. And, at the same time, an additional constant is introduced into the equation between [illegible] and y, which enables us to give to v the value required in the statement of the problem. Hence, when u is to be made a maximum or minimum, while the value of v is constant, we must make $\delta (u + av) = 0$, proceeding in the same manner as in the simpler cases. And, if it were required that u should be a maximum or minimum, the values of the functions v [illegible] being constant, the same reasoning would shew that [illegible] $+ av + bw) = 0$; and so for any number of [illegible].

[illegible] Taking the instance above, the area $= \int_x y$: the length $\int_x$ [illegible] $+p^2$ [illegible]

$$\therefore u + [illegible] = \int_x (y + a\sqrt{1+p^2});\ \therefore\ V = y + a\sqrt{1+p^2}:$$

$$\text{and } [illegible]\, p + C, \text{ or } y + a\sqrt{1+p^2} = \frac{ap^2}{\sqrt{1+p^2}} + C;$$

$$\frac{a}{\sqrt{1+p^2}} = C - y;\ \frac{dx}{dy} = \frac{1}{p} = \frac{C-y}{\sqrt{a^2 - \overline{C-y}|^2}};$$

$$x = C' - \sqrt{a^2 - (C-y)^2};\ \therefore\ (x - C')^2 + (y - C)^2 = a^2,$$

the equation to a circular arc. If the limits be fixed, the values of the constant must be determined so as to make the length of the arc equal to the given length, and to make it pass through the two given points. If the limits be not fixed, suppose the first and last ordinates AM, BN, fig. 4, to be given; then, since $\delta y_{,} = 0$, $\delta y_{,,} = 0$, the equation for the limits reduces itself to

$$(V_{,,} - P_{,,}p_{,,})\,\delta x_{,,} - (V_{,} - P_{,}p_{,})\,\delta x_{,} = 0;$$

from which

$$V_{,,} - P_{,,}p_{,,} = 0,\ V_{,} - P_{,}p_{,} = 0.$$

Since $V - Pp = C$, these equations are satisfied by making $C = 0$;

$$\therefore\ (x - C')^2 + y^2 = a^2:$$

that is, when the lengths of the ordinates AM, BN, and the arc AB are given, the area $AMNB$ is a maximum if AB be a circular arc, whose center C is in the line MN. If AM, BN, each $= 0$, MN is the chord of AB: and, it appears that the curve which, with a given length, contains between its chord and its arc the greatest area, is the semi-circle.

22. Given the length of a curve, to find its form, [illegible] center of gravity may be the lowest possible. Let [illegible] $= b$; then, the depth of the center of gravity $= \frac{1}{b}\int_x$ [illegible] the length $= \int_x \sqrt{1+p^2}$; hence

$$V = a\sqrt{1+p^2} + \frac{y\sqrt{1+p^2}}{b};$$

and making $V = Pp + C$, $\dfrac{a + \frac{y}{b}}{\sqrt{1+p^2}} = C$,

whence $x = bC \cdot \log \{y + ba + \sqrt{(\overline{y+ba}|^2 - b^2C^2)}\} + C'$;

the equation to the catenary. If the extreme points be not fixed, but move on curves, it will be found that the catenary will cut the curves at right angles.

23. Given the surface of a solid of revolution, to find its nature, that the solid content may be a maximum. Let x be measured along the axis of revolution;

the surface $= 2\pi \int_x \sqrt{1+p^2}$; the solidity $= \pi \int_x y^2$;

$\therefore V = y^2 + ay\sqrt{1+p^2}$; $V = Pp + C$ gives $\dfrac{ay}{\sqrt{1+p^2}} = C - y^2$,

$$\text{whence } \frac{dx}{dy} = \frac{C - y^2}{\sqrt{a^2y^2 - (C-y^2)^2}}.$$

[illegible] and last ordinates of the generating curve, and the distance [illegible]tween them, be fixed, this equation (supposing it integrated) is sufficient; the three constants which the integral equation [illegible]ll contain must be determined so as to make the first [illegible] ordinates, and the surface, equal to the given [illegible] the distance between the first and last ordinates [illegible] $\delta y_, = 0$, $\delta y_{,,} = 0$, and the equation for the [illegible]

$[illegible]) \delta x_{,,} - (V - P_, p_,) \delta x_, = 0$, or $C\delta x_{,,} - C\delta x_, = 0$;

[illegible] no relation between $\delta x_{,,}$ and $\delta x_,$, $C = 0$.

[illegible], $\dfrac{dx}{dy} = \dfrac{-y^2}{\sqrt{a^2y^2 - y^4}} = \dfrac{-y}{\sqrt{a^2 - y^2}}$;

[illegible] $+ C' = \sqrt{a^2 - y^2}$,

the equation to a circle, whose center is in the line of abscissæ: and the solid is, therefore, a portion of a sphere included between two planes perpendicular to a diameter. If the first and last ordinates be 0, the solid is a whole sphere.

24. Given the whole surface of a solid of revolution (including the circular ends), to find its form, that the solid content may be a maximum. The solidity $=\pi\int_x y^2$: the surface

$$=2\pi\int_x y\sqrt{1+p^2}+\pi(y_,^2+y_{,,}^2);$$

hence, we must make

$$\delta\left(\int_x y^2+a\int_x y\sqrt{1+p^2}+\frac{a}{2}(y_,^2+y_{,,}^2)\right)=0.$$

The part under the sign of integration will be the same as in the last problem, and the equation derived from it will be the same, or

$$\frac{dx}{dy}=\frac{C-y^2}{\sqrt{a^2y^2-(C-y^2)^2}}.$$

The part depending on the variation of the limits, besides the usual terms

$$(V_{,,}-P_{,,}p_{,,})\,\delta x_{,,}-(V_,-P_,p_,)\,\delta x_,+P_{,,}\delta y_{,,}-\text{]}$$

must have the terms which express the variation

$$\text{that is,}\quad a(y_,\delta y_,+y_{,,}\delta y_{,,}).$$

Making the whole $=0$,

$$C\delta x_{,,}-C\delta x_,+ay_{,,}\left(1+\frac{p_{,,}}{\sqrt{1+p_{,,}}}\right.$$

$$+ay_,\left(1-\frac{p_,}{\sqrt{1+p_,^2}}\right.$$

The two first coefficients shew that $C=0$, and that the solid is part of a sphere; the third and fourth require that $y_{\prime\prime}=0$, $y_{\prime}=0$; or that $\frac{1}{p_{\prime\prime}}=0$, $\frac{1}{p_{\prime}}=0$, which agree with the former; hence, the solid is an entire sphere.

25. Required the curve of quickest descent from one given point to another given point, the length of the curve being given.

$$\text{Here } V=\frac{a\sqrt{1+p^2}}{\sqrt{y}}+\sqrt{1+p^2},$$

$$\text{and } V=Pp+C \text{ gives } \frac{a}{\sqrt{y}}+1=C\sqrt{1+p^2}$$

the differential equation to the curve. The constants must be determined so as to make the length of the curve equal to the given quantity, and to make the curve pass through both points.

26. Given the mass of a solid of revolution, required its form, that the attraction upon a point in the axis may be a maximum. Let the solid be divided into slices by planes perpendicular to the axis of revolution, then, since the attraction of a circle, whose thickness is h and radius y, upon a point at the distance x from its center, is ultimately

$$2\pi h\left(1-\frac{x}{\sqrt{x^2+y^2}}\right),$$

...e solid will

$$=2\pi\int\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)$$

$\pi\int y^2$. Hence,

$$V=1-\frac{x}{\sqrt{x^2+y^2}}+ay^2,$$

the equ... $-\frac{d(P)}{dx}+$ &c. $=0$ is reduced to $N=$

$$\frac{xy}{(x^2+y^2)^{\frac{1}{2}}} + 2ay = 0, \quad \text{or } x + 2a(x^2+y^2)^{\frac{1}{2}} = 0,$$

the equation to the curve by the revolution of which the solid is generated. If the first and last values of x be given, a must be determined, so that the included solid = the given solidity. If the first and last values of x are indeterminate, the equation of the limits is

$$V_{\prime\prime}\delta x_{\prime\prime} - V_{\prime}\delta x_{\prime} = 0, \quad \text{or } V_{\prime\prime} = 0,\ V_{\prime} = 0;\ \therefore\ y_{\prime\prime} = 0,\ y_{\prime} = 0;$$

and the solid must be that generated by the revolution of the whole curve, whose equation is

$$x + 2a(x^2+y^2)^{\frac{1}{2}} = 0,$$

a being determined so that the whole solidity = the given solidity.

27. The rules and examples above will serve to elucidate all the cases that commonly occur; and the same principles are easily be extended to more difficult problems. For investigations of the cases where u is given by double integration, or by the solution of a differential equation, or where V is a function of two or more independent variables, or where V depends upon another integral, the reader is referred to Lacroix, *Traité du Calcul Différentiel, et du Calcul Intégral*, Tome 2me, or to Woodhouse's *Isoperimetrical Problems*.